出版前言

2011年12月11日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰100周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共10卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

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Section 1

Interorbital Acceleration

$\xi(\bullet) = \sqrt{m}$

Interorbital Acceleration

Let g = gravitational acceleration at starting radius r_0.
$(\ )_0$ = initial

$$\frac{d^2r}{dt^2} = R + r\left(\frac{d\theta}{dt}\right)^2 - g\left(\frac{r_0}{r}\right)^2$$

$$r\frac{d^2\theta}{dt^2} = \Theta - 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)$$

1) Case $\Theta = 0$, radial thrust, R = constant

$$r\frac{d^2\theta}{dt^2} + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) = 0$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

$$r^2\frac{d\theta}{dt} = r_0^2\left(\frac{d\theta}{dt}\right)_0 \qquad \frac{d\theta}{dt} = \left(\frac{r_0}{r}\right)^2\left(\frac{d\theta}{dt}\right)_0$$

$$\frac{d^2r}{dt^2} = R + \frac{r_0^4\left(\frac{d\theta}{dt}\right)_0^2}{r^3} - \frac{g r_0^2}{r^2}$$

$$\int_0^t M\Theta\, dt = M\int_0^t \Theta\, dt - \frac{dM}{dt}\int_0^t \Theta\, dt$$

If the start is made from a satellite,

$$r_0\left(\frac{d\theta}{dt}\right)_0^2 = g$$

$$\frac{d^2r}{dt^2} = R + g\frac{r_0^3}{r^3} - g\frac{r_0^2}{r^2} \qquad \frac{dr}{dt} = v_r$$

$$\frac{1}{2}\frac{dv_r^2}{dr} = R + g r_0^2\left\{\frac{r_0}{r^3} - \frac{1}{r^2}\right\}$$

Since $v_r = 0$ at $r = r_0$, $t = 0$,

$$\frac{1}{2} v_r^2 = R(r - r_0) + g r_0^2 \left\{ \frac{1}{2} \frac{1}{r_0} - \frac{1}{2} \frac{r_0}{r^2} + \frac{1}{r} - \frac{1}{r_0} \right\}$$

$$\boxed{v_r^2 = 2R(r - r_0) + g r_0^2 \left\{ \frac{2}{r} - \frac{r_0}{r^2} - \frac{1}{r_0} \right\}}$$

Total energy at r

$$E = \left\{ \frac{1}{2} \left[v_r^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 \right] - \frac{g r_0^2}{r} \right\} M$$

$$= \left\{ \frac{1}{2} \left[v_r^2 + \frac{1}{r^2} r_0^3 g \right] - \frac{g r_0^2}{r} \right\} M$$

$$= \left\{ \frac{1}{2} \left[2R(r - r_0) + g r_0^2 \left(\frac{2}{r} - \frac{1}{r_0} \right) \right] - \frac{g r_0^2}{r} \right\} M$$

$$= \left\{ R(r - r_0) - \frac{1}{2} g r_0 \right\} M$$

$$\Delta E = \left\{ R(r - r_0) - \frac{1}{2} g r_0 \right\} M - \left\{ \frac{1}{2} g r_0 - g r_0 \right\} M_0$$

$$= M R (r - r_0) + \frac{1}{2} g r_0 (M_0 - M)$$

thrust $\quad RM = -c \dfrac{dM}{dt}$

$$\frac{dr}{dt} = \left\{ 2Rr - (2Rr_0 + g r_0) + 2 g r_0^2 \frac{1}{r} - g r_0^3 \frac{1}{r^2} \right\}^{1/2}$$

$$dt = \frac{dr}{\sqrt{2Rr - (2Rr_0 + g r_0) + 2 g r_0^2 \frac{1}{r} - g r_0^3 \frac{1}{r^2}}}$$

$$dt = -\frac{c}{R}\frac{dM}{M}$$

$$\boxed{t = \frac{c}{R}\log_e\frac{M_0}{M}} \qquad R = ng$$

$$t = \int_{r_0} \frac{dr}{\sqrt{2Rr - (2Rr_0 + gr_0) + 2gr_0^2\frac{1}{r} - gr_0^3\frac{1}{r^2}}}$$

$$= \frac{1}{\sqrt{g}}\int_{r_0}\frac{r\,dr}{\sqrt{2nr^3 - (2nr_0 + r_0)r^2 + 2r_0^2 r - r_0^3}}$$

$$= \frac{1}{\sqrt{g}}\int_{r_0}\frac{r\,dr}{\sqrt{r-r_0}\sqrt{2nr^2 - r_0 r + r_0^2}}$$

$$2nr^2 - r_0 r + r_0^2 > 0.$$

$$2nr^2 > r_0(r - r_0)$$

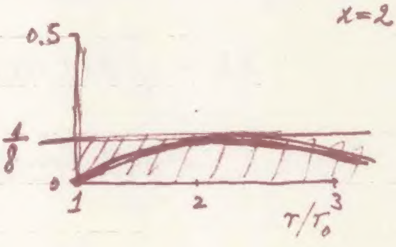
$$2n > \frac{r_0}{r} - \frac{r_0^2}{r^2}$$

$$n > \frac{1}{2}\frac{r_0}{r}\left(1 - \frac{r_0}{r}\right)$$

$$\frac{1}{x}\left(1-\frac{1}{x}\right)$$

$$-\frac{1}{x^2} + 2\frac{1}{x^3} = 0$$

$$x = 2$$



$$R(r - r_0) - \frac{1}{2}gr_0 = 0$$

$$Rr = (R + \frac{1}{2}g)r_0$$

$$r = \left(1 + \frac{1}{2n}\right)r_0$$

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$\eta = 1+\xi$

$$t\sqrt{\frac{g}{r_0}} = \int_1^{1+\frac{1}{2n}} \frac{\eta\, d\eta}{\sqrt{\eta-1}\sqrt{2n\eta^2-\eta+1}}$$

$$= \int_0^{\frac{1}{2n}} \frac{(1+\xi)\,d\xi}{\sqrt{\xi}\sqrt{2n\xi^2+4n\xi+2n-1-\xi+1}}$$

$$= \int_0^{\frac{1}{2n}} \frac{(1+\xi)\,d\xi}{\sqrt{\xi}\sqrt{2n\xi^2+(4n-1)\xi+2n}}$$

$$= \int_0^{\frac{1}{2n}} \frac{d\xi}{\xi^{\frac{1}{2}}\left[2n\xi^2+(4n-1)\xi+2n\right]^{\frac{1}{2}}} + \int_0^{\frac{1}{2n}} \frac{\xi^{\frac{1}{2}}\,d\xi}{\left[2n\xi^2+(4n-1)\xi+2n\right]^{\frac{1}{2}}}$$

$=$

$$\int_0^t R\,dt = \int$$

$2n-1 \quad 2n\xi^2 + (4n-1)\xi + 2n$

$/$

$$\frac{2\,\xi^{\frac{1}{2}}}{[\quad]^{1/2}} + \int \frac{\xi^{\frac{1}{2}}\cdot\left[4n\xi+(4n-1)\right]d\xi}{[\quad]^{3/2}}$$

s

160

24

5

$$\int_0^t M\Theta\,dt = M\int_0^t \Theta\,dt + \int_0^t -\frac{dM}{dt}dt\int_0^t \Theta\,dt'$$

$$-\frac{dM}{dt} = \frac{\Theta}{c}M$$

$$M\int_0^t \Theta\,dt + \frac{1}{c}\int_0^t \left(\Theta\,dt\int_0^t \Theta\,dt'\right)$$

$$\frac{1}{c}\int \frac{\log r}{r}\,dr$$

$$= \frac{1}{2}(\log r)^2$$

$1-\varepsilon+\varepsilon^2$

$1-2\varepsilon+3\varepsilon^2$

$-\frac{3}{2}+$

$-\frac{1}{2}\varepsilon^2$

$$\frac{1}{2}\frac{dv_r^2}{dr} = g\frac{r_0^3}{r^3} - g\frac{r_0^2}{r^2}$$

$$\frac{1}{2}v_r^2 = \frac{gr_0^3}{2}\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right) - gr_0^2\left(\frac{1}{r_0}-\frac{1}{r}\right)$$

$$= gr_0\left[\frac{1}{2}\left(1-\frac{r_0^2}{r^2}\right) - \left(1-\frac{r_0}{r}\right)\right]$$

$$\left(1-\frac{r_0}{r}\right)\left[\frac{1}{2}+\frac{1}{2}\frac{r_0}{r}-1\right]$$

$$\frac{1}{2}-\frac{1}{2}\frac{r_0^2}{r^2}-1+\frac{r_0}{r}$$

$$-\frac{1}{2}\left(1-\frac{r_0}{r}\right)^2$$

2) <u>Case $R=0$</u>

$$\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = \Theta r$$

$$r^2 \frac{d\theta}{dt} = r_0^2 \left(\frac{d\theta}{dt}\right)_0 + \int_0^t \Theta r \, dt \qquad \int_0^r \frac{\Theta r}{\sqrt{r}} dr$$

$$\frac{d^2 r}{dt^2} = \frac{1}{r^3}\left\{ r_0^2 \left(\frac{d\theta}{dt}\right)_0 + \int_0^t \Theta r \, dt \right\}^2 - g \frac{r_0^2}{r^2}$$

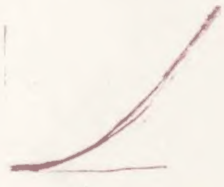
$$\boxed{\frac{d^2 r}{dt^2} = g \frac{r_0^2}{r^2}\left(\frac{r_0}{r} - 1\right) + 2 \frac{r_0^2 \left(\frac{d\theta}{dt}\right)_0}{r^3} \int_0^t \Theta r \, dt + \frac{1}{r^3}\left[\int_0^t \Theta r \, dt\right]^2}$$

At $t=0$, $\frac{d^2 r}{dt^2} = 0$, $r = r_0$, $\frac{dr}{dt} = 0$.

$$\left(r^3 \frac{d^2 r}{dt^2} + g r_0^2 r\right)^{\frac{1}{2}} = r_0^2 \left(\frac{d\theta}{dt}\right)_0 + \int_0^t \Theta r \, dt$$

$$\boxed{\Theta r = \frac{1}{2} \frac{3r^2 \frac{dr}{dt}\frac{d^2 r}{dt^2} + r^3 \frac{d^3 r}{dt^3} + g r_0^2 \frac{dr}{dt}}{\left(r^3 \frac{d^2 r}{dt^2} + g r_0^2 r\right)^{\frac{1}{2}}}}$$

$$\Theta r = \frac{1}{2} \frac{\frac{1}{2} 3r^2 \frac{dv_r^2}{dr} + \frac{1}{2} r^3 \frac{d^2 v_r^2}{dr^2} + g r_0^2}{\left(r^3 \frac{1}{2}\frac{dv_r^2}{dr} + g r_0^2 r\right)^{\frac{1}{2}}} v_r$$



$$r = \frac{\alpha t^3}{1+\beta t^2} + r_0$$

$$v_r \frac{d}{dr}\left(r^3 \frac{1}{2} \frac{dv_r^2}{dr}\right)$$

$$mg r_0 \eta = \frac{1}{2} \frac{(r_0^4/T^3)\left[3\eta^2 \frac{d\eta}{d\tau}\frac{d^2\eta}{d\tau^2} + \eta^3 \frac{d^3\eta}{d\tau^3}\right] + g \frac{r_0^3}{T} \frac{d\eta}{d\tau}}{\left[\frac{r_0^4}{T^2} \eta^3 \frac{d^2\eta}{d\tau^2} + g r_0^3 \eta\right]^{\frac{1}{2}}}$$

$4\alpha - 3\beta = \alpha - \beta$

$3\alpha = 2\beta$

$\beta = 1.5\alpha$

$$\frac{r_0^4}{T^3} = \frac{g r_0^3}{T}$$

$$m g r_0 \eta = \frac{1}{2}\frac{r_0^2}{T^2}\frac{X}{\lambda^2}$$

$$\frac{3\eta^2 \frac{d\eta}{dt}\frac{d^2\eta}{dt^2} + \eta^3 \frac{d^3\eta}{dt^3} + \frac{d\eta}{dt}}{\left[\eta^3 \frac{d^2\eta}{dt^2} + \eta\right]^{\frac{1}{2}}}$$

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$4\alpha - 2\beta = \alpha$

$$\frac{r_0}{T^2} = g \qquad T^2 = \frac{r_0}{g} \qquad T^2$$

$$\eta = \frac{\xi}{\sqrt{m}}$$

$1.5\alpha + 1 = \alpha - \beta$

$\beta = 0.5\alpha$

$$m g \frac{r_0 T^2}{r_0^2}$$

ξ

$$t = \frac{\tau}{m^{3/4}}$$

$2\alpha + 1 = 0$

$\alpha = -\frac{1}{2}$

$$\boxed{2m\eta\left[\eta^3 \frac{d^2\eta}{dt^2} + \eta\right]^{\frac{1}{2}} = 3\eta^2 \frac{d\eta}{dt}\frac{d^2\eta}{dt^2} + \eta^3 \frac{d^3\eta}{dt^3} + \frac{d\eta}{dt}}$$

$$\int_0^t \Theta\, dt = F \qquad 2m = \frac{d^3\eta}{dt^3}$$

$$\int_0^t \Theta r\, dt = Fr - \int_{r_0}^{r} F\, dr$$

$$-1 + \frac{1}{x}$$

$$\frac{1}{2} - \frac{1}{2x^2}$$

$$\frac{1}{2} - \frac{1}{2x^2} + \frac{1}{x}$$

$$-\frac{1}{2}\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)$$

$$\left(1 - \frac{1}{x}\right)^2$$

$$\frac{d^3\eta}{dt^3} \frac{d\eta}{dt}\frac{d}{d\eta}\left(\frac{1}{2}\frac{d\eta^2}{d\eta}\right)$$

$$\left[\eta^3 \frac{1}{x}\frac{dx}{dr} + \frac{\eta}{x}\right]^{\frac{1}{2}} =$$

$$\eta = 1 + \frac{2m}{6}t^3 + a_4 t^4 + a_5 t^5 + \ldots$$

$$\frac{d\eta}{dt} = \frac{2m}{2}t^2$$

$$\frac{d^2\eta}{dt^2} = 2mt$$

$$\frac{(-2)(-3)(-4)}{2\cdot 3}$$

$1-\varepsilon$

$-1+\varepsilon-\varepsilon^2 \cdots$

$-gr_0(\varepsilon-\varepsilon^2 =)$

$(1+\varepsilon)^{-2} = 1-2\varepsilon+3\varepsilon^2 \cdots$

$$-gr_0\left(1-\frac{r_0}{r}\right)$$

$$+\frac{1}{2}gr_0\left(1-\frac{r_0^2}{r^2}\right)$$

$2K^2-2K+1=0$

$K^2-K+\frac{1}{2}=0$

$K=-\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{1}{2}}$

$$+gr_0\left[\varepsilon-\frac{3}{2}\varepsilon^2\right]$$

$$-gr_0\frac{1}{2}\varepsilon^2$$

$$K^2\cancel{\varepsilon}+\left[2K-1-2K^2\right]\cancel{\varepsilon}+(K-1)^2\cancel{\varepsilon} = 0$$

$2K-1-2K+1=0$

$$\frac{1}{2}v_r^2 = gr_0\left[K^2\log\frac{r}{r_0}+\{2K(1-K)-1\}\left(1-\frac{r_0}{r}\right)+\frac{1}{2}(K-1)^2\left(1-\frac{r_0^2}{r^2}\right)\right]$$

$$\int_0^t \Theta r\,dt = \int_0^r \frac{\Theta r}{v_r}\,dr$$

Let $\quad \dfrac{\Theta r}{v_r} = K\sqrt{g r_0}$

$$\int_0^t \Theta r\,dt = K\sqrt{g r_0}\,(r - r_0)$$

$$\frac{d^2 r}{dt^2} = \frac{1}{2}\frac{dv_r^2}{dr} = \frac{g r_0^2}{r^3}\left(r_0 - r\right) + 2\,\frac{K g r_0^2}{r^3}(r - r_0)$$
$$+ \frac{K^2 g r_0}{r^3}(r - r_0)^2$$

$$= g r_0^2 (2K-1)\,\frac{r - r_0}{r^3} + \frac{K^2 g r_0}{r^3}\left(r^2 - 2 r r_0 + r_0^2\right)$$

$$= \frac{K^2 g r_0}{r} + \left[g r_0^2 (2K-1) - 2K^2 g r_0^2\right]\frac{1}{r^2} + \left[K^2 g r_0^3 - g r_0^3 (2K-1)\right]\frac{1}{r^3}$$

$$2K - 2K^2 - 1 \qquad\qquad (K-1)^2$$

$$2K(1-K) - 1$$

$$-\left[1 + 2K(K-1)\right]$$

$$\frac{1}{2}v_r^2 = K^2 g r_0 \log\frac{r}{r_0} + g r_0^2\left[2K - 1 - 2K^2\right]\left[\frac{1}{r_0} - \frac{1}{r}\right]$$
$$+ \frac{1}{2}(K-1)^2 g r_0^3\left[\frac{1}{r_0^2} - \frac{1}{r^2}\right]$$

$$\int_0^t \Theta\,dt = \int_0^t \frac{\Theta}{v_r}\,dr = \int_0^r \frac{dr}{r}\,K\sqrt{g r_0}$$
$$= K\sqrt{g r_0}\,\log\frac{r}{r_0}$$

$\mathcal{E}$

$$\frac{1}{2} r^2 \left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}\frac{1}{r^2}\left[r_0^2\left(\frac{d\theta}{dt}\right)_0 + K\sqrt{gr_0}\,(r-r_0)\right]^2$$

$$= \frac{1}{2}\frac{1}{r^2}\left[r_0\sqrt{gr_0} + K\sqrt{gr_0}\,(r-r_0)\right]^2$$

$$= \frac{1}{2}\frac{r_0^2}{r^2} gr_0 \left[1 + K\left(\frac{r}{r_0}-1\right)\right]^2$$

$K\varepsilon = \frac{1}{2}$

$$= \frac{1}{2}\frac{r_0^2}{r^2} gr_0 \left[(1-K) + K\frac{r}{r_0}\right]^2$$

$$= \frac{1}{2}\frac{r_0^2}{r^2} gr_0 \left[(1-K)^2 + 2K(1-K)\frac{r}{r_0} + K^2\frac{r^2}{r_0^2}\right]$$

$$\frac{1}{2}\left[v_r^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]$$

$K\varepsilon - \frac{1}{2} = 0$

$$= \frac{1}{2}(K-1)^2 gr_0 + K^2 gr_0 \ln\left(\frac{r}{r_0}\right) + gr_0\left[2K(1-K) - 1\right]$$

$$+ \frac{1}{2}K^2 gr_0 + gr_0^2\frac{1}{r}\left[-\cancel{2}K + 1 + \cancel{2}K^2 + \cancel{2}K - \cancel{2}K^2\right]$$

$\frac{r_0}{r} = 1-\varepsilon$

$$= \left(\cancel{K^2} - K + \tfrac{1}{2} + 2K - 2K^2 - \tfrac{1}{2}\right) gr_0 + K^2 gr_0 \ln\left(\frac{r}{r_0}\right) -$$

$$E = \left(K - K^2 - \tfrac{1}{2}\right)\cancel{gr_0} + K^2 \cancel{gr_0} \ln\left(\frac{r}{r_0}\right) - (\cancel{\quad})\ln\left(\frac{r_0}{r}\right)$$

$K(K-1)$ $= 0$

$$- \cancel{g\frac{r_0^2}{r}} \qquad K \qquad \cancel{K - K^2} - \frac{1}{2} + K^2\varepsilon + \cancel{(K^2 - K)}$$

$$-(K^2 - K)\varepsilon = 0$$

$$-\frac{1}{2} + K^2\left(\varepsilon - \frac{\varepsilon^2}{2}\right) - (K^2 - K)(\varepsilon - \varepsilon^2) = 0$$

$$-\frac{1}{2} + K^2\left[\varepsilon - \frac{\varepsilon^2}{2} - \varepsilon + \varepsilon^2\right] + K(\varepsilon - \varepsilon^2) = 0$$

$$-\frac{1}{2} + K^2\frac{\varepsilon^2}{2} + K(\varepsilon - \varepsilon^2) = 0$$

$$\cancel{\tfrac{1}{2}}(K\varepsilon)^2 + 2(K\varepsilon)(1-\varepsilon) - \overset{1}{\cancel{\tfrac{1}{2}}} = 0$$

$$K\varepsilon = 1-\varepsilon \pm \sqrt{(1-\varepsilon)^2 + 1}$$

$$= 1 + \sqrt{2}$$

$$K^2 \log\frac{r}{r_0} = \frac{1}{2}$$

$$K \log\frac{r}{r_0} = \frac{1}{2K}$$

$$\Theta M = -c\,\frac{dM}{dt}$$

$$M\,\Theta = K\sqrt{q r_0}\,\frac{1}{r}\,\frac{dr}{\cancel{dt}}\,M = -c\,\frac{dM}{\cancel{dt}}$$

$$\frac{1}{c}\,K\sqrt{q r_0}\,\frac{dr}{r} = -\,\frac{dM}{M}$$

$$\frac{1}{c}\,K\sqrt{q r_0}\,\lg\frac{r}{r_0} = \lg\frac{M_0}{M}$$

$$\frac{M_0}{M} = \left(\frac{r}{r_0}\right)^{\frac{1}{c}K\sqrt{q r_0}}$$

$$\Theta = \frac{F}{M} = \frac{F}{M_0 - \frac{F}{c}t}$$

Interorbital Acceleration

g = gravitational acceleration at starting radius r_0
$(\ )_0$ = initial

$$\frac{d^2r}{dt^2} = R + r\left(\frac{d\theta}{dt}\right)^2 - g\left(\frac{r_0}{r}\right)^2$$

$$r\frac{d^2\theta}{dt^2} + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) = \Theta$$

1) Case $\Theta = 0$, R = constant

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

$$r^2\frac{d\theta}{dt} = r_0^2\left(\frac{d\theta}{dt}\right)_0, \qquad \frac{d\theta}{dt} = \left(\frac{r_0}{r}\right)^2\left(\frac{d\theta}{dt}\right)_0$$

$$\frac{d^2r}{dt^2} = R + \frac{r_0^4\left(\frac{d\theta}{dt}\right)_0^2}{r^3} - \frac{g r_0^2}{r^2}$$

If the start is made from a satellite,

$$r_0\left(\frac{d\theta}{dt}\right)_0^2 = g$$

So
$$\frac{d^2r}{dt^2} = R + \frac{g r_0^3}{r^3} - \frac{g r_0^2}{r^2}$$

Let
$$v_r = \frac{dr}{dt}$$

$$\frac{1}{2}\frac{dv_r^2}{dr} = R + \frac{g r_0^3}{r^3} - \frac{g r_0^2}{r^2}$$

Since $v_r = 0$ at $r = r_0$, $t = 0$.

$$\frac{1}{2}v_r^2 = R(r-r_0) + \frac{1}{2}g r_0^3\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - g r_0^2\left(\frac{1}{r_0} - \frac{1}{r}\right)$$

$$= R(r-r_0) + \frac{1}{2}g r_0^2\left[\frac{1}{r_0} - \frac{r_0}{r^2} - \frac{2}{r_0} + \frac{2}{r}\right]$$

$$\frac{1}{2}v_r^2 = R(r-r_0) + \frac{1}{2}g r_0^2\left[-\frac{1}{r_0} + \frac{2}{r} - \frac{r_0}{r^2}\right]$$

Total energy at r, per unit mass

$$\frac{1}{2}(v_r^2 + v_\theta^2) - \frac{g r_0^2}{r}$$

$$= R(r-r_0) + \frac{1}{2}g r_0^2\left[-\frac{1}{r_0} + \cancel{\frac{2}{r}} - \cancel{\frac{r_0}{r^2}}\right] + \frac{1}{2}\cancel{\frac{r_0^3 g}{r^2}} - \cancel{\frac{g r_0^2}{r}}$$

$$= R(r-r_0) - \frac{1}{2}g r_0$$

At the end of the powered flight, $(\ )_1$

$$R(r_1 - r_0) - \frac{1}{2}g r_0 = 0$$

So if $R = ng$,

$$n(r_1 - r_0) - \frac{1}{2}r_0 = 0$$

$$n\left(\frac{r_1}{r_0} - 1\right) = \frac{1}{2} \qquad \boxed{\frac{r_1}{r_0} = 1 + \frac{1}{2n}}$$

The instantaneous mass M, exhaust velocity c,

$$MR = -c\frac{dM}{dt} = Mgn$$

$$dt = -\frac{c}{gn}\frac{dM}{M}, \qquad t_1 = \frac{c}{gn}\log\frac{M_0}{M_1}$$

$$\boxed{\log\frac{M_0}{M_1} = \frac{gn}{c}t_1}$$

$$\frac{dr}{dt} = \left\{ 2ng(r-r_0) + g r_0^2 \left[-\frac{1}{r_0} + \frac{2}{r} - \frac{r_0}{r^2} \right] \right\}^{\frac{1}{2}}$$

$$= \sqrt{g} \left\{ (r-r_0) \left[2n - r_0 \left(\frac{1}{r} - \frac{r_0}{r^2} \right) \right] \right\}^{\frac{1}{2}}$$

$$t_1 = \frac{1}{\sqrt{g}} \int_{r_0}^{(1+\frac{1}{2n})r_0} \frac{dr}{\sqrt{(r-r_0)\left[2n - \left(\frac{r_0}{r} - \frac{r_0^2}{r^2}\right)\right]}}$$

$$= \sqrt{\frac{r_0}{g}} \int_1^{1+\frac{1}{2n}} \frac{d\eta}{\sqrt{(\eta-1)\left[2n - \frac{1}{\eta} + \frac{1}{\eta^2}\right]}}$$

$$\boxed{t_1 = \sqrt{\frac{r_0}{g}} \int_1^{1+\frac{1}{2n}} \frac{\eta\, d\eta}{\sqrt{(\eta-1)\left[2n\eta^2 - \eta + 1\right]}}}$$

Let $\eta = 1 + \xi$

$$t_1 \sqrt{\frac{g}{r_0}} = \int_0^{\frac{1}{2n}} \frac{(1+\xi)\, d\xi}{\sqrt{\xi\left[2n\xi^2 + (4n-1)\xi + 2n\right]}}$$

$$\xi = \frac{1}{2n} s$$

$$t_1 \sqrt{\frac{g}{r_0}} = \int_0^1 \frac{\left(1 + \frac{s}{2n}\right) \frac{ds}{2n}}{\sqrt{\frac{1}{2n} s \left[\frac{1}{2n} s^2 + \frac{4n-1}{2n} s + 2n\right]}}$$

$$= \frac{1}{2n} \int_0^1 \frac{(2n+s)\, ds}{\sqrt{s\left[s^2 + (4n-1)s + 4n^2\right]}}$$

When $n \gg 1$,

$$t_1\sqrt{\frac{g}{r_0}} = \frac{1}{2n}\int_0^1 \frac{\left(1+\frac{s}{2n}\right)ds}{\sqrt{s\left[1+\frac{4n-1}{4n^2}s+\frac{s^2}{4n^2}\right]}}$$

$$= \frac{1}{2n}\int_0^1 \frac{\left(1+\frac{s}{2n}\right)ds}{\sqrt{s}}\left\{1-\frac{4n-1}{8n^2}s-\frac{s^2}{8n^2}+\frac{3}{8}\frac{(4n-1)^2}{16n^4}s^2\cdots\right\}$$

$$= \frac{1}{2n}\int_0^1 \frac{ds}{\sqrt{s}}\left\{1-\frac{4n-1}{4n}\left(\frac{s}{2n}\right)+\left(\frac{3}{8}\frac{(4n-1)^2}{4n^2}-\frac{1}{2}\right)\left(\frac{s}{2n}\right)^2\cdots\right.$$
$$\left.+\left(\frac{s}{2n}\right)-\frac{4n-1}{4n}\left(\frac{s}{2n}\right)^2\right\}$$

$$= \frac{1}{2n}\int_0^1 \frac{ds}{\sqrt{s}}\left\{1+\frac{1}{4n}\left(\frac{s}{2n}\right)+\left[\frac{3}{2}\left(1-\frac{1}{4n}\right)^2-\frac{1}{2}-\left(1-\frac{1}{4n}\right)\right]\left(\frac{s}{2n}\right)^2\right\}$$

$$= \frac{1}{2n}\left\{2+\frac{2}{3}\frac{1}{4n}\frac{1}{2n}+\frac{2}{5}\left[\frac{3}{2}\left(1-\frac{1}{4n}\right)^2-\frac{1}{2}-\left(1-\frac{1}{4n}\right)\right]\frac{1}{4n^2}\cdots\right\}$$

$$= \frac{1}{n}\left\{1+\frac{1}{24n^2}+\frac{1}{20n^2}\left[\cancel{\tfrac{3}{2}}-\frac{3}{4n}+\frac{3}{32n^2}-\cancel{\tfrac{1}{2}}-\cancel{1}+\frac{1}{4n}\right]\cdots\right\}$$

$$= \frac{1}{n}\left\{1+\frac{1}{24n^2}+\frac{1}{20n^2}\left[-\frac{1}{2n}+\frac{3}{32n^2}\cdots\right]\cdots\right\}$$

$$t_1\sqrt{\frac{g}{r_0}} = \frac{1}{n}\left\{1+\frac{1}{24n^2}-\frac{1}{40n^3}\cdots\right\}$$

$$\boxed{\log\frac{M_0}{M_1} = \frac{\sqrt{g r_0}}{c}\left\{1+\frac{1}{24n^2}-\frac{1}{40n^3}\cdots\right\}} \qquad n \gg 1$$

g_0 surface g, R = radius of earth

$$g = g_0\frac{R^2}{r_0^2} \qquad g r_0 = g_0\frac{R^2}{r_0} = \left(\frac{R}{r_0}\right)\frac{1}{2}(2g_0R)$$

$$\boxed{\log \frac{M_0}{M_1} = \frac{1}{\sqrt{2\left(\frac{r_0}{R}\right)}}\,\frac{V}{c}\left\{1 + \frac{1}{24n^2} - \frac{1}{4n^3} \cdots\right\}} \qquad n \gg 1$$

$$\boxed{V = \sqrt{2 g_0 R} = \text{escape velocity}}$$

$$v_{r_1}^2 = 2ngr_0\left(\frac{r_1}{r_0} - 1\right) + gr_0\left[-1 + 2\frac{r_0}{r_1} - \frac{r_0^2}{r_1^2}\right]$$

$$= gr_0\left[2n \cdot \frac{1}{2n} - \left(1 - \frac{1}{1+\frac{1}{2n}}\right)^2\right]$$

$$= gr_0\left[1 - 1 + \frac{2}{1+\frac{1}{2n}} - \frac{1}{\left(1+\frac{1}{2n}\right)^2}\right]$$

$$= gr_0\,\frac{1}{\left(1+\frac{1}{2n}\right)^2}\left[2 + \frac{1}{n} - 1\right] = gr_0\,\frac{1+\frac{1}{n}}{\left(1+\frac{1}{2n}\right)^2}$$

$$\boxed{v_{r_1} = \frac{1}{\sqrt{2\frac{r_0}{R}}}\,V\,\frac{\sqrt{1+\frac{1}{n}}}{1+\frac{1}{2n}}}$$

$$v_{\theta_1} = r_1\left(\frac{d\theta}{dt}\right)_1 = r_1\,\frac{r_0^2}{r_1^2}\left(\frac{d\theta}{dt}\right)_0 = r_1\,\frac{r_0^{3/2}}{r_1^2}\sqrt{g}$$

$$\boxed{v_{\theta_1} = \left(\frac{r_0}{r_1}\right)\sqrt{gr_0} = \frac{1}{1+\frac{1}{2n}}\,\frac{1}{\sqrt{2\left(\frac{r_0}{R}\right)}}\,V}$$

$$\boxed{\left(\frac{v_r}{v_\theta}\right)_1 = \sqrt{1+\frac{1}{n}}}$$

In general,

$$t_1\sqrt{\frac{g}{r_0}}\cdot\sqrt{2n} = \int_0^{\frac{1}{2n}} \frac{(1+\xi)\,d\xi}{\sqrt{\xi\left[\xi^2+\frac{4n-1}{2n}\xi+1\right]}}$$

According to Magnus, p. 108, $\quad \alpha=0, \quad 2b=-\frac{4n-1}{2n}, \quad c=1$

$$H=\sqrt{\alpha^2-2b\alpha+1} = 1, \quad k=\sqrt{\frac{1-(1-\frac{1}{4n})}{2}} = \sqrt{\frac{1}{8n}}, \quad k'=\sqrt{1-\frac{1}{8n}}$$

$$\int_0^{\frac{1}{2n}} \frac{(1+\xi)\,d\xi}{\sqrt{\xi\left[\xi^2+\frac{4n-1}{2n}\xi+1\right]}} = 2\int_{\frac{2n-1}{2n+1}}^{1} \frac{dz}{(1+z)\sqrt{(1-z^2)\left[\left(1-\frac{1}{8n}\right)+\frac{1}{8n}z^2\right]}}$$

$$z=\frac{1-\xi}{1+\xi}, \qquad (1+\xi)z = 1-\xi \qquad (1+z)\xi = 1-z$$

$$\xi = \frac{1-z}{1+z}$$

$$1+\xi = 1+\frac{1-z}{1+z} = \frac{2}{1+z}$$

$$d\xi = \frac{(-1-z-1+z)\,dz}{(1+z)^2} = -\frac{2}{(1+z)^2}\,dz$$

$$\xi^2+\frac{4n-1}{2n}\xi+1 = \frac{1-\cancel{2z}+z^2+\left(\frac{4n-1}{2n}\right)(1-z^2)+1+\cancel{2z}+z^2}{(1+z)^2}$$

$$= \frac{\left(4-\frac{1}{2n}\right)+\frac{1}{2n}z^2}{(1+z)^2}$$

$$\frac{1-\frac{1}{2n}}{1+\frac{1}{2n}} = \frac{2n-1}{2n+1}$$

when $\underline{n=\frac{1}{8},}$ $\qquad t_1\sqrt{\frac{g}{r_0}}\,\frac{1}{2} = 2\int_{-3/5}^{1}\frac{dz}{|z|(1+z)\sqrt{1-z^2}} = \infty$

for $n > \frac{1}{8}$,

$$I(n) = \int_{\frac{2n-1}{2n+1}}^{1} \frac{dz}{(1+z)(1-z^2)^{1/2}\left[1-\frac{1}{8n}(1-z^2)\right]^{1/2}} = \frac{\sqrt{2(n+1)}}{2n+1} + F\left(\frac{1}{\sqrt{8n}}, \cos^{-1}\frac{2n-1}{2n+1}\right)$$
$$- E\left(\frac{1}{\sqrt{8n}}, \cos^{-1}\frac{2n-1}{2n+1}\right)$$

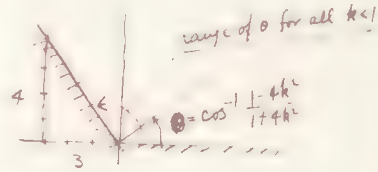
where $F(k,\varphi) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, $\quad E(k,\varphi) = \int_0^{\varphi}\sqrt{1-k^2\sin^2\theta}\, d\theta$

Proof:

$$I(n) = \int_{\frac{1-4k^2}{1+4k^2}}^{1} \frac{dz}{(1+z)(1-z^2)^{1/2}\left[1-k^2(1-z^2)\right]^{1/2}} \qquad \left(k^2 = \frac{1}{8n} < 1\right)$$

(lower limit $\frac{1-4k^2}{1+4k^2} = \frac{2}{1+4k^2} - 1 > (-\frac{3}{5})$, but always < 1.)

by substitution $z = \cos\theta$



$$I(n) = \int_0^{\cos^{-1}\frac{1-4k^2}{1+4k^2}} \frac{d\theta}{(1+\cos\theta)\sqrt{1-k^2\sin^2\theta}}$$

$$= \int_0^{\cos^{-1}\frac{1-4k^2}{1+4k^2}} \frac{(1-\cos\theta)\,d\theta}{\sin^2\theta\sqrt{1-k^2\sin^2\theta}}$$

$$= \lim_{\epsilon\to 0}\left\{\int_{\epsilon}^{\varphi}\frac{d\theta}{\sin^2\theta\sqrt{1-k^2\sin^2\theta}} - \int_{\epsilon}^{\varphi}\frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{1-k^2\sin^2\theta}}\right\}$$

$$\left/\ \begin{aligned}\varphi &= \cos^{-1}\frac{1-4k^2}{1+4k^2}\\ &= \sin^{-1}\frac{4k}{1+4k^2}\end{aligned}\right)$$

$1+4k^2$, φ, $1-4k^2$, $4k$

$$= \lim_{\epsilon\to 0}\left\{I_1(k,\epsilon) - I_2(k,\epsilon)\right\}$$

For $I_1(k,\epsilon)$, integrating by parts once

$$I_1(k,\epsilon) = \int_{\epsilon}^{\varphi}\frac{d(-\cot\theta)}{\sqrt{1-k^2\sin^2\theta}} = -\frac{\cot\theta}{\sqrt{1-k^2\sin^2\theta}}\Bigg|_{\epsilon}^{\varphi} + k^2\int_{\epsilon}^{\varphi}\frac{\cos^2\theta\, d\theta}{(1-k^2\sin^2\theta)^{3/2}}$$

$$= \frac{\cot\epsilon}{\sqrt{1-k^2\sin^2\epsilon}} - \frac{1-4k^2}{4k}\frac{1+4k^2}{\sqrt{1+8k^2}} + k^2\int_{\epsilon}^{\varphi}\frac{\cos^2\theta\, d\theta}{(1-k^2\sin^2\theta)^{3/2}}$$

$I_2(k,\epsilon)$ is an elementary integral.

$$I_2(k,\epsilon) = \int_\epsilon^\varphi \frac{\cos\theta \, d\theta}{\sin^2\theta\sqrt{1-k^2\sin^2\theta}} = -\frac{\sqrt{1-k^2\sin^2\theta}}{\sin\theta}\Bigg|_\epsilon^\varphi$$

$$= \frac{\sqrt{1-k^2\sin^2\epsilon}}{\sin\epsilon} - \frac{\sqrt{1+8k^2}}{4k}$$

Hence

$$I(n) = \lim_{\epsilon\to 0}\left\{\frac{(k^2-\frac{1}{2})\epsilon^2 + O(\epsilon^3)}{\sin\epsilon\sqrt{1-k^2\sin^2\epsilon}}\right\} + 2k\frac{1+2k^2}{\sqrt{1+8k^2}} + k^2\int_0^\varphi \frac{\cos^2\theta\, d\theta}{(1-k^2\sin^2\theta)^{3/2}}$$

$$= 2k\frac{1+2k^2}{\sqrt{1+8k^2}} + F(k,\varphi) - E(k,\varphi) + k^2\frac{\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

(Jahnke & Emde p.56, or Magnus p.113)

$$= 2k\frac{1+2k^2}{\sqrt{1+8k^2}} + k^2\frac{4k}{1+4k^2}\cdot\frac{1-4k^2}{\cancel{1+4k^2}}\,\frac{\cancel{1+4k^2}}{\sqrt{1+8k^2}} + F(k,\varphi) - E(k,\varphi)$$

$$= \frac{2k}{1+4k^2}\sqrt{1+8k^2} + F(k,\varphi) - E(k,\varphi)$$

Substituting back $k^2 = \frac{1}{8n}$, $\varphi = \cos^{-1}\frac{2n-1}{2n+1}$, we have

$$\boxed{I(n) = \frac{\sqrt{2}\sqrt{n+1}}{2n+1} + F\left(\frac{1}{\sqrt{8n}}, \cos^{-1}\frac{2n-1}{2n+1}\right) - E\left(\frac{1}{\sqrt{8n}}, \cos^{-1}\frac{2n-1}{2n+1}\right)} \quad \text{---(1)}$$

As a check, take $n=\frac{1}{2}$. F & E becomes complete elliptic integral. Equ. (1) gives

$$I(\tfrac{1}{2}) = \frac{\sqrt{3}}{2} + K(\tfrac{1}{2}) - E(\tfrac{1}{2})$$

$$= \frac{\sqrt{3}}{2} + 1.6857 - 1.4675 \qquad \text{(Jahnke \& Emde)}$$

$$= 1.0842$$

Now we check this result by numerical integration.

From the original integral

$$I(\tfrac{1}{2}) = \int_0^1 \frac{2\,dz}{(1+z)^{3/2}(1-z)^{1/2}\,(3+z^2)^{1/2}}$$

$$= -\frac{4\sqrt{1-z}}{(1+z)^{3/2}\,(3+z^2)^{1/2}}\Bigg|_0^1 + 4\int_0^1 \sqrt{1-z}\;\frac{d}{dz}\left(\frac{1}{(1+z)^{3/2}(3+z^2)^{1/2}}\right)dz$$

$$= \frac{4}{\sqrt{3}} - 2\int_0^1 \frac{5z^2+2z+9}{(1+z)^2(3+z^2)}\sqrt{\frac{1-z}{(1+z)(3+z^2)}}\;dz.$$

Use Simpson's Rule.

z	$f(z) = \dfrac{5z^2+2z+9}{(1+z)^2(3+z^2)}\sqrt{\dfrac{1-z}{(1+z)(3+z^2)}}$	weighting factor	Product
0	$\sqrt{3} = 1.732$	1	1.7321
$\frac{1}{4}$	0.906	4	3.624
$\frac{1}{2}$	0.491	2	.982
$\frac{3}{4}$	0.244	4	.976
1	0	1	0
		sum	7.314

$$\therefore\; I(\tfrac{1}{2}) = \frac{4}{\sqrt{3}} - \left(\frac{7.314}{12}\right)2 = 2.304 - 1.219 = 1.085$$

which agrees with formula (1)

$$t_1 \sqrt{\frac{g}{r_0}} \sqrt{2n} = 2\, I(n)$$

$$\log \frac{M_0}{M_1} = \frac{gn}{c} t_1 = \frac{gn}{c} \sqrt{\frac{r_0}{g}} \frac{1}{\sqrt{2n}}\, 2\, I(n)$$

$$= \frac{\sqrt{g r_0}}{c} \sqrt{2n}\, I(n)$$

$$\boxed{\log \frac{M_0}{M_1} = \frac{1}{\sqrt{2(R_0/R)}} \frac{V}{c} \left\{ \sqrt{2n}\, I(n) \right\}}$$

$$\sqrt{2n}\, I(n) = \frac{2\sqrt{n(n+1)}}{2n+1} + \sqrt{2n} \left\{ F\left(\frac{1}{\sqrt{8n}}, \cos^{-1}\frac{2n-1}{2n+1}\right) - E\left(\frac{1}{\sqrt{8n}}, \cos^{-1}\frac{2n-1}{2n+1}\right) \right\}$$

$\underline{n = \frac{1}{4},}$ $\quad \frac{1}{\sqrt{8n}} = \frac{1}{\sqrt{2}}, \quad \alpha = 45^\circ, \quad \frac{2n-1}{2n+1} = \frac{\frac{1}{2}-1}{\frac{1}{2}+1} = -\frac{1}{3}$

$$\cos^{-1}\left(-\frac{1}{3}\right) = \pi - \cos^{-1}\left(\frac{1}{3}\right)$$

$$= \pi - 70.53^\circ$$

$$F\left(\frac{1}{\sqrt{8n}}, \cos^{-1}\frac{2n-1}{2n+1}\right) = 2 \times 1.8541 - 1.3821 = 2.3261$$

$$E\left(\frac{1}{\sqrt{8n}}, \cos^{-1}\frac{2n-1}{2n+1}\right) = 2 \times 1.3506 - 1.1059 = 1.5953$$

$$\sqrt{2n}\, I(n) = \frac{\frac{1}{2}\sqrt{1 \times 5}}{\frac{3}{2}} + \frac{1}{\sqrt{2}}\left(2.3261 - 1.5953\right)$$

$$= \frac{1}{3}\sqrt{5} + \frac{1}{\sqrt{2}} \times 0.7308 = 0.745 + 0.517 = \underline{1.262}$$

$\underline{n = \frac{1}{2}}$ $\qquad \sqrt{2n}\, I(n) = \underline{1.0842}$

$\underline{n = \frac{1}{6}}$

$$\frac{1}{\sqrt{8n}} = \frac{\sqrt{3}}{2}, \qquad \frac{1}{8n} = \frac{3}{4}, \qquad n = \frac{1}{8}\frac{4}{3} = \frac{1}{6}$$

$$\alpha = \sin^{-1}\frac{\sqrt{3}}{2} = 60^\circ$$

$$\frac{2n-1}{2n+1} = \frac{\frac{1}{3}-1}{\frac{1}{3}+1} = -\frac{2}{4} = -\frac{1}{2}, \qquad \varphi = \pi - 60^\circ$$

$$F\left(\frac{1}{\sqrt{8n}},\ \cos^{-1}\frac{2n-1}{2n+1}\right) = 2\times 2.1565 - 1.2125 = 3.1005$$

$$E \qquad = 2\times 1.2411 - 0.9184 = 1.5038$$

$$\sqrt{2n}\,I(n) = \frac{\frac{1}{3}\sqrt{1.7}}{\frac{4}{3}} + \frac{1}{\sqrt{3}}(3.1005 - 1.5038)$$

$$= 0.661 + 0.923 = \underline{1.584}$$

$\underline{n = 1.068}$

$$\alpha = 20^\circ, \qquad \frac{1}{8n} = \sin^2 20^\circ = 0.34202^2$$

$$n = \frac{1}{2\times 0.68404^2} = 1.068$$

$$\cos^{-1}\frac{2n-1}{2n+1} = \cos^{-1}\frac{1.136}{3.136} = \cos^{-1}0.3622 = 68.77^\circ$$

$$F = \underset{142}{1.2127} + 0.0184\times 0.77 = 1.2269$$

$$E = \underset{127}{1.1619} + 0.0165\times 0.77 = 1.1746$$

$$\sqrt{2n}\,I(n) = \frac{2\sqrt{1.068\times 2.068}}{3.136} + \sqrt{2.136}\times 0.0523$$

$$= \underset{77}{0.948} + 0.077 = \underline{1.025}$$

$$n = \frac{1}{8 \times 0.98481^2} \qquad \alpha = 80^\circ, \qquad \sin\alpha = 0.98481$$

$$= 0.1289 \qquad \frac{1}{8n} = 0.98481^2 \qquad \frac{1}{n} = 8 \times 0.98481^2$$

$$\frac{2n-1}{2n+1} = \frac{\frac{1}{4\times 0.98481^2} - 1}{\frac{1}{4\times 0.98481^2}+1} = -\frac{4\times 0.98481^2 - 1}{4\times 0.98481^2+1}$$

$$= -\frac{1.96962^2-1}{1.96962^2+1} = -\frac{0.96962 \times 2.96962}{1+1.96962^2} = -0.59011$$

1152
867
285

4.87740 $\varphi = \pi - 53.83^\circ$

$$F = 2\times 3.1534 - \begin{array}{r}1.0867\\236\\\hline 1.1103\end{array} = \begin{array}{r}6.3068\\1.1103\\\hline 5.1965\end{array}$$

3.87940
4.87740
8.75880

$$E = 2\times 1.0401 - \begin{array}{r}0.8037\\88\\\hline 0.8125\end{array} = \begin{array}{r}2.0802\\.8125\\\hline 1.2677\end{array}$$

$n = \frac{1}{\alpha}$

$$\sqrt{2n}\, I(n) = \cancel{\frac{1}{4\times 0.98481^2}} \sqrt{1+1.96962^2\times 2}\ \frac{1}{1+1.96962^2}$$

$\frac{2}{\alpha}\sqrt{1+\alpha}$

$\frac{2}{\alpha}+1$

$$+ \frac{1}{1.96962}\left(5.1965 - 1.2677\right) \qquad \begin{array}{r}1.2677\\\hline 3.9288\end{array}$$

$$= \frac{\sqrt{8.75880}}{4.87740} + \frac{3.9288}{1.96962}$$

$$= 0.606 + 1.994 = \underline{2.600}$$

606
00

Section 2

Emissivity of Diatomic Gases at Low Pressures

Emissivity of Diatomic Gases at Low Pressures

H. S. Tsien

I. Introduction

Emissivity calculations for diatomic gases from spectroscopic data were developed recently by S. S. Penner (Ref. 1). His method is based upon the use of an average absorption coefficient for the entire fundamental and higher vibration- rotation bands. The method is thus effective when there are extensive overlapping and broadening of the spectral lines, and hence is accurate for gases at high total pressures and temperatures. At low pressures, the lines do not overlap and a different approach to the problem should be made. Penner and M. H. Ostrander (Ref. 2) have computed the emissivity of carbon monoxide for the case of non-overlapping lines by a numerical procedure, using spectroscopic data obtained recently by Penner and D. Weber (Ref. 3). The results are in excellent agreement with the emissivity determined experimentally by W. Ullrich and H. C. Hottel (Ref. 4). The amount of numerical work involved is however rather heavy. It is the purpose of the present paper to develop an approximate but convenient formula for calculating the emissivity of diatomic gases for the case of non-overlapping lines.

II. Formulation of the Problem

If T is the temperature, θ the characteristic temperature, ν the wave number, ν^* the characteristic wave number, P_ν the spectral absorption coefficient at ν, p the partial pressure of the emitting gas, and L the optical path length, then the emissivity ε of the gas under the specified conditions is

$$\varepsilon = \int_0^\infty \frac{\nu^3 \{1 - e^{-P_\nu p L}\}}{e^{(\theta/T)\frac{\nu}{\nu^*}} - 1} d\nu \Bigg/ \int_0^\infty \frac{\nu^3 \, d\nu}{e^{(\theta/T)\frac{\nu}{\nu^*}} - 1} \qquad (1)$$

If only the fundamental vibrational-rotational band is considered, the absorption coefficient P_ν is given by

$$P_\nu = \frac{b}{\pi} \sum_{j=1}^{\infty} \left[\frac{S^{0\to1}_{j\to j-1}}{(\nu-\nu^{0\to1}_{j\to j-1})^2+b^2} + \frac{S^{0\to1}_{j-1\to j}}{(\nu-\nu^{0\to1}_{j-1\to j})^2+b^2} \right] \qquad (2)$$

where b the half-width of the spectral lines, and S's are the integrated absorptions for the lines centering on the wave numbers corresponding to the indicated transitions. The S's can be computed in turn by using the results of J. R. Oppenheimer (Ref. 5) as

$$S^{0\to1}_{j\to j-1} = \frac{N_T \varepsilon^2 \pi}{3\mu c^2 Q} \, \frac{\nu^{0\to1}_{j\to j-1}}{\nu^*} \, j \, e^{-\frac{E_{0,j}}{kT}} \, F\cdot G \qquad (3)$$

and

$$S^{0\to1}_{j-1\to j} = \frac{N_T \varepsilon^2 \pi}{3\mu c^2 Q} \, \frac{\nu^{0\to1}_{j-1\to j}}{\nu^*} \, j \, e^{-\frac{E_{0,j-1}}{kT}} \, F'G' \qquad (4)$$

where N_T is the number of emitting molecules at temperature T per unit volume per unit pressure, ε the effective charge, μ the reduced mass, c the velocity of light, Q the complete internal partition function. E's are the internal energy levels given by

$$E_{v,j} = k\theta \left[v - xv(v-1) + \gamma j(j+1)\{1 - 4\gamma^2 j(j+1) - \delta\cdot v\} \right] \qquad (5)$$

where x, γ, δ are molecular constants in their standard notation. These constants are non-dimensional and are small.
The F's and G's are

$$F(j,\gamma) = 1 + 8\gamma j\left(1 + \frac{5\gamma j}{4} - \frac{3\gamma}{4}\right)$$
$$F'(j,\gamma) = 1 - 8\gamma j\left(1 - \frac{5\gamma j}{4} - \frac{3\gamma}{4}\right) = F(-j,\gamma) \qquad (6)$$

and

$$G = 1 - \exp\left\{ -\left(\frac{hc}{kT}\right) \nu^{0\to1}_{j\to j-1} \right\}$$
$$G' = 1 - \exp\left\{ -\left(\frac{hc}{kT}\right) \nu^{0\to1}_{j-1\to j} \right\} \qquad (7)$$

the complete internal partition function can be written as

$$Q_{i} = \frac{1}{\gamma \frac{\theta}{T}(1-e^{-\theta/T})}\left[1+\gamma\left(\frac{1}{3}\frac{\theta}{T}+8\frac{T}{\theta}\right)+\frac{\delta}{e^{\theta/T}-1}+\frac{2x\frac{\theta}{T}}{(e^{\theta/T}-1)^2}\right] \qquad (8)$$

If the fundamental vibrational-rotational band gives the main contribution to the emissivity of the gas, the above equations give the necessary information to calculate approximately the emissivity ε.

III. Approximate Solution

The numerical work in carrying out the computation indicated in the previous section is very heavy. A short formula, however, can be developed: First of all, when the lines are separated from each other, each line can be considered alone, independent of others. Furthermore, the value of the factor outside of the bracket in the numerator of Eq. (7) can be approximated by its value at the center of each line. Thus according S. S. Penner (Ref. 6)

$$\varepsilon = \frac{15}{\pi^4}\left(\frac{\theta}{T}\right)^4 \sum_{j=1}^{\infty}\left[\frac{\left(\nu^{0\to1}_{j\to j-1}/\nu^*\right)^3}{e^{\left(\frac{\theta}{T}\right)\left(\nu^{0\to1}_{j\to j-1}/\nu^*\right)}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j\to j-1}pL}\right)d\left(\frac{\nu}{\nu^*}\right)\right.$$

$$\left.+\frac{\left(\nu^{0\to1}_{j-1\to j}/\nu^*\right)^3}{e^{\left(\frac{\theta}{T}\right)\left(\nu^{0\to1}_{j-1\to j}/\nu^*\right)}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j-1\to j}pL}\right)d\left(\frac{\nu}{\nu^*}\right)\right] \qquad (9)$$

where the Ps are the absorption coefficient due to the particular line with transitions as indicated. The integrals can be easily evaluated (Ref. 6) and are given by the modified Bessel functions I_0 and I_1:

$$\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j\to j-1}pL}\right)d\left(\frac{\nu}{\nu^*}\right) = 2\pi\left(\frac{b}{\nu^*}\right)\xi_j e^{-\xi_j}\left[I_0(\xi_j)+I_1(\xi_j)\right] \qquad (10)$$

and $$\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to 1}_{j-1\to j}\,pL}\right)d\left(\frac{\nu}{\nu^*}\right)=2\pi\left(\frac{b}{\nu^*}\right)\eta_j\,e^{-\eta_j}\left[I_0(\eta_j)+I_1(\eta_j)\right] \qquad (11)$$

where $$\xi_j = S^{0\to 1}_{j\to j-1}\,/2\pi b \qquad (12)$$

and $$\eta_j = S^{0\to 1}_{j-1\to j}\,/2\pi b \qquad (13)$$

A further approximation can now be made: the magnitude of ξ's and η's are generally quite large if the product pL of pressure and optical path length is of the order of unity. Therefore the asymptotic values of the Bessel functions can be used. Then

$$\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to 1}_{j\to j-1}\,pL}\right)d\left(\frac{\nu}{\nu^*}\right)\cong 2\sqrt{\frac{b\,S^{0\to 1}_{j\to j-1}}{\nu^{*2}}} \qquad (14)$$

and $$\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to 1}_{j-1\to j}\,pL}\right)d\left(\frac{\nu}{\nu^*}\right)\cong 2\sqrt{\frac{b\,S^{0\to 1}_{j-1\to j}}{\nu^{*2}}} \qquad (15)$$

By substituting Eqs. (14) and (15) into (9), the emissivity is calculated as a sum over j.

To carry out the sum over j, one can use the Euler-Maclaurin summation formula (Ref. 7), which evaluates the sum by an integral. First, due to the smallness of γ, δ, the following expansions, including terms up to the square of γj, are appropriate:

$$\frac{\nu^{0\to 1}_{j\to j-1}}{\nu^*}=\left(1-2\gamma j-\left(\frac{\delta}{\gamma}\right)\gamma^2 j^2\right) \qquad (16)$$

$$\sqrt{F(j)} = 1+2\gamma j-\frac{1}{2}\gamma^2 j^2 \qquad (17)$$

$$\frac{\sqrt{G}}{e^{\left(\frac{\theta}{T}\right)\left(\nu^{0\to 1}_{j\to j-1}/\nu^*\right)}-1}=e^{-\left(\frac{\theta}{T}\right)\left(\nu^{0\to 1}_{j\to j-1}/\nu^*\right)}\left\{1-e^{-\left(\frac{\theta}{T}\right)\left(\nu^{0\to 1}_{j\to j-1}/\nu^*\right)}\right\}^{-\frac{1}{2}} \qquad (18)$$

$$= \frac{e^{-\frac{\theta}{T}}}{(1-e^{-\theta/T})^{1/2}}\left[1+\left(\frac{\theta}{T}\right)\left\{2+\frac{e^{-\frac{\theta}{T}}}{(1-e^{-\theta/T})}\right\}\gamma j+\left\{\left(\frac{\theta}{T}\right)\left(\frac{\xi}{\gamma}+2\frac{\theta}{T}\right)+\frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{(1-e^{-\theta/T})}\left(3\frac{\theta}{T}+\frac{\xi}{2\gamma}\right)+\frac{3}{2}\frac{\left(\frac{\theta}{T}\right)^2e^{-2\theta/T}}{(1-e^{-\theta/T})^2}\right\}\gamma^2 j^2\right] \tag{19}$$

and

$$e^{-\frac{E_{0,j}}{2kT}} = e^{-\frac{\gamma}{2}\frac{\theta}{T}j^2}\left[1-\frac{1}{2}\left(\frac{\theta}{T}\right)\delta j+\frac{1}{8}\left(\frac{\theta}{T}\right)^2\delta^2 j^2\right] \tag{20}$$

The corresponding quantities for the transitions $j-1\to j$ can be very easily obtained from Eqs. (16) to (20) by replacing j with $-j$. Because of this property of asymmetry ~~property~~, the sum of terms from the transition $j\to j-1$ and the transition $j-1\to j$ for every j is a function of only j^2. Thus

$$\frac{\left(\nu^{0\to1}_{j\to j-1}/\nu^*\right)^3}{e^{\frac{\theta}{T}\left(\nu^{0\to1}_{j\to j-1}/\nu^*\right)}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j\to j-1}\rho L}\right)d\left(\frac{\nu}{\nu^*}\right)+\frac{\left(\nu^{0\to1}_{j-1\to j}/\nu^*\right)^3}{e^{\frac{\theta}{T}\left(\nu^{0\to1}_{j-1\to j}/\nu^*\right)}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j-1\to j}\rho L}\right)d\left(\frac{\nu}{\nu^*}\right)$$

$$= 4\left(\frac{\theta}{T}\right)e^{-\left(\frac{\theta}{T}\right)}f\left(\gamma,\delta,\xi,\frac{\theta}{T}\right)\sqrt{\left(\frac{\Delta\nu b}{\nu^*}\right)\left(\frac{A\rho L}{\nu^*}\right)}\,\sqrt{j}\,e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2}\left[1+g\left(\gamma,\delta,\frac{\theta}{T}\right)\gamma^2 j^2\right] \tag{21}$$

where

$$A = \frac{N_T\,\mathcal{E}^2\pi}{3\mu c^2}\,\frac{T}{\theta} \tag{22}$$

A is thus a constant independent of temperature and pressure. The f function is simply deduced from the partition function Q as given by Eq. (8):

$$f = 1-\gamma\left(\frac{1}{6}\frac{\theta}{T}+4\frac{T}{\theta}\right)-\frac{\delta/2}{e^{\theta/T}-1}-\frac{x\left(\frac{\theta}{T}\right)}{(e^{\theta/T}-1)^2} \tag{23}$$

Therefore f is a quantity close to unity. The function g is computed from the relations given in Eqs. (16) to (20). It is

$$g = \frac{3}{2}\left\{\frac{\left(\frac{\theta}{T}\right)e^{-\theta/T}}{1-e^{-\theta/T}}\right\}^2+\frac{\left(\frac{\theta}{T}\right)e^{-\theta/T}}{1-e^{-\theta/T}}\left\{\frac{5}{2}\left(\frac{\theta}{T}\right)+\frac{\xi}{2\gamma}-5\right\}+\left(\frac{\theta}{T}\right)\left\{\frac{\xi}{\gamma}-\frac{15}{2}+\frac{9}{8}\frac{\theta}{T}\right\}+\frac{11}{4}-\frac{7}{2}\frac{\xi}{\gamma} \tag{24}$$

The Euler-Maclaurin formula can be now employed to evaluate the sum in the emissivity ε. The resulting integral over j extends from $-\infty$ to ∞. But this range can be made to be from 0 to ∞ by simply deduct

the approximate value of the integral from 0 to 1 from the extended integral. Thus

$$2\sum_{j=1}^{\infty}\sqrt{j}\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2}\left[1+g\left(\gamma,\delta,\tfrac{\theta}{T}\right)\gamma^2 j^2\right] \cong 2\int_1^{\infty}\sqrt{j}\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2}\left[1+g\gamma^2 j^2\right]dj + \frac{11}{12}$$

$$\cong -\frac{5}{12} + 2\int_0^{\infty}\sqrt{j}\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2}\left[1+g\gamma^2 j^2\right]dj$$

$$= -\frac{5}{12} + \Gamma\left(\tfrac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/4}\left[1+\frac{3}{2}\frac{\gamma T}{\theta}g\right]$$

The $\Gamma\left(\frac{3}{4}\right)$ has the numerical value of 1.225...

Finally then the expression for emissivity for the case of non-overlapping lines is

$$\mathcal{E} = \frac{30}{\pi^4}\left(\frac{\theta}{T}\right)^5 e^{-\theta/T} f\left(\gamma,\delta,x,\tfrac{\theta}{T}\right)\left[\Gamma\left(\tfrac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/4}\left\{1+\frac{3}{2}\frac{\gamma T}{\theta}g\left(\gamma,\delta,\tfrac{\theta}{T}\right)\right\}-\frac{5}{12}\right] \cdot \sqrt{\left(\frac{\gamma b}{\nu^*}\right)\left(\frac{A p L}{\nu^*}\right)} \qquad (25)$$

where f and g are functions given previously in Eqs. (23) and (24).

Since the value of f is nearly unity and the factor before g in Eq. (25) is small, a good approximate equation for the emissivity is

$$\mathcal{E} \cong \frac{30}{\pi^4}\left(\frac{\theta}{T}\right)^5 e^{-\theta/T}\,\Gamma\left(\tfrac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/4}\sqrt{\left(\frac{\gamma b}{\nu^*}\right)\left(\frac{A p L}{\nu^*}\right)} \qquad (26)$$

IV. Application to Carbon Monoxide

For carbon monoxide, the molecular constants are

$\theta = 3066.9\,^{\circ}K$

$\nu^* = 2142.3\ cm^{-1}$

$\gamma = 0.000895$

$\delta = 0.0091$

$$\alpha = 0.00620$$

The value of A computed from the measurements of Penner and Weber (Ref.3) is

$$A = 24.16 \text{ atm}^{-1} \text{ cm}^{-2}$$

They have also determined b to be 0.077 cm^{-1} at one atmosphere of total pressure. According to the approximate equation (26), the emissivity at $T = 300°K$ and a total pressure of one atmosphere

$$\varepsilon = 1.630 \times 10^{-3} \sqrt{pL} \quad (27)$$

where pL is in atm-cm. By using the more exact equation (25), the emissivity is

$$\varepsilon = 1.640 \times 10^{-3} \sqrt{pL} \quad (28)$$

The difference between the approximate value and the more exact value is quite small. The comparison between the computed emissivity and the measurements of Ullrich and Hottel (Ref.4) is shown in Fig.1. The agreement is quite satisfactory up to pL of approximately 10.

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Overlapping Case

Emissivity of Diatomic Gases

$$P_\nu p L = \frac{\alpha N_T \varepsilon^2 p L}{3\mu c^2 Q_{njm} \nu^{*2}} \sum_{j=1}^{\infty} \left[\frac{j(1-2\gamma j)\, e^{-\gamma\frac{h}{T} j(j+1)}}{\left\{\left(\frac{\nu}{\nu^0}-1\right)+2\gamma j\right\}^2 + \left(\frac{\alpha}{\nu^0}\right)^2} + \frac{j(1+2\gamma j)\, e^{-\gamma\frac{h}{T} j(j-1)}}{\left\{\left(\frac{\nu}{\nu^0}-1\right)-2\gamma j\right\}^2 + \left(\frac{\alpha}{\nu^0}\right)^2} \right]$$

Take the first sum, $\quad f(j) = \dfrac{j(1-2\gamma j)\, e^{-\gamma\frac{h}{T} j(j+1)}}{\left\{\left(\frac{\nu}{\nu^0}-1\right)+2\gamma j\right\}^2 + \left(\frac{\alpha}{\nu^0}\right)^2}$

$$\frac{\partial f}{\partial j} = \frac{e^{-\gamma\frac{h}{T} j(j+1)}}{\left\{\left(\frac{\nu}{\nu^0}-1\right)+2\gamma j\right\}^2 + \left(\frac{\alpha}{\nu^0}\right)^2} \left\{ (1-4\gamma j) - j(1-2\gamma j)\gamma\frac{h}{T}(2j+1) - \frac{j(1-2\gamma j)\, 2\left\{\left(\frac{\nu}{\nu^0}-1\right)+2\gamma j\right\} 2\gamma}{\left\{\left(\frac{\nu}{\nu^0}-1\right)+2\gamma j\right\}^2 + \left(\frac{\alpha}{\nu^0}\right)^2} \right\}$$

Therefore it seems that in order for the Euler-Maclaurin formula to hold,

$$\gamma\left(\frac{\nu^0}{\alpha}\right)^2 < 1.$$

Then $\quad P_\nu p L = \dfrac{\alpha N_T \varepsilon^2 p L}{3\mu c^2 Q_{njm} \nu^{*2}} \left\{ \displaystyle\int_0^\infty \left[\frac{j(1-2\gamma j)\, e^{-\gamma\frac{h}{T} j(j+1)}}{\left\{\left(\frac{\nu}{\nu^0}-1\right)+2\gamma j\right\}^2 + \left(\frac{\alpha}{\nu^0}\right)^2} + \frac{j(1+2\gamma j)\, e^{-\gamma\frac{h}{T} j(j-1)}}{\left\{\left(\frac{\nu}{\nu^0}-1\right)-2\gamma j\right\}^2 + \left(\frac{\alpha}{\nu^0}\right)^2} \right] dj - \frac{1}{6}\, \frac{1}{\left(\frac{\nu}{\nu^0}-1\right)^2 + \left(\frac{\alpha}{\nu^0}\right)^2} \right\}$

The integrals are

$$= \int_0^\infty \frac{\left[\cos \gamma\frac{h}{T} j + 2\gamma j \sin \gamma\frac{h}{T} j\right]\left[\left\{\left(\frac{\nu}{\nu^0}-1\right)^2 + \left(\frac{\alpha}{\nu^0}\right)^2\right\} + 4\gamma^2 j^2\right] + 4\left[\sin \gamma\frac{h}{T} j + 2\gamma j \cos \gamma\frac{h}{T} j\right]\left(\frac{\nu}{\nu^0}-1\right)\gamma j}{\left\{\left(\frac{\nu}{\nu^0}-1\right)^2 + \left(\frac{\alpha}{\nu^0}\right)^2\right\}^2 - 8\left\{\left(\frac{\nu}{\nu^0}-1\right)^2 - \left(\frac{\alpha}{\nu^0}\right)^2\right\}\gamma^2 j^2 + 16\gamma^4 j^4}\, e^{-\frac{\beta h}{T} j^2}\, 2j\, dj$$

$$\cong \int_0^\infty \frac{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}+\left\{4+4\left(\frac{\nu}{\nu^*}-1\right)\left(2+\frac{\theta}{T}\right)+\frac{\theta}{T}\left(2+\frac{1}{2}\frac{\theta}{T}\right)\left[\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]\right\}\gamma^2\zeta^2}{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}^2-8\left\{\left(\frac{\nu}{\nu^*}-1\right)^2-\left(\frac{\alpha}{\nu^*}\right)^2\right\}\gamma^2\zeta^2}\, e^{-\gamma\frac{\theta}{T}\zeta^2}\, d(\zeta^2)$$

Let $\quad \gamma\frac{\theta}{T}\zeta^2 = \eta$

$$\gamma^2\zeta^2 = \gamma\left(\frac{T}{\theta}\right)\eta$$

$$= \frac{T}{\gamma\theta}\,\frac{1}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}\int_0^\infty\left[1+\left\{\frac{4+4\left(\frac{\nu}{\nu^*}-1\right)\left(2+\frac{\theta}{T}\right)}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}+\frac{\theta}{T}\left(2+\frac{1}{2}\frac{\theta}{T}\right)+8\frac{\left(\frac{\nu}{\nu^*}-1\right)^2-\left(\frac{\alpha}{\nu^*}\right)^2}{\left[\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]^2}\right\}\gamma\frac{T}{\theta}\eta\right] e^{-\eta}\,d\eta$$

$$= \frac{\frac{T}{\gamma\theta}}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}\left[1+\gamma\frac{T}{\theta}\left\{\frac{12+4\left(\frac{\nu}{\nu^*}-1\right)\left(2+\frac{\theta}{T}\right)}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}+\frac{\theta}{T}\left(2+\frac{1}{2}\frac{\theta}{T}\right)-16\frac{\left(\frac{\alpha}{\nu^*}\right)^2}{\left[\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]^2}\right\}\right]$$

$$\boxed{\text{the sum} = \frac{\frac{T}{\gamma\theta}+\left(2\frac{T}{\theta}+\frac{1}{3}\right)}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}+4\left(\frac{T}{\theta}\right)^2\left[\frac{3+\left(\frac{\nu}{\nu^*}-1\right)\left(2+\frac{\theta}{T}\right)}{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}^2}-\frac{4\left(\frac{\alpha}{\nu^*}\right)^2}{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}^3}\right]}$$

$$P_\nu pL = \frac{\alpha N_0 \varepsilon^2 pL}{3\mu c^2\nu^{*2}}\left(\frac{\theta}{T}\right)\left(1-e^{-\theta/T}\right)\left[\frac{1+\gamma\left(2+\frac{1}{3}\frac{\theta}{T}\right)}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}\right.$$

$$\left.+4\gamma\frac{T}{\theta}\left\{\frac{3+\left(\frac{\nu}{\nu^*}-1\right)\left(2+\frac{\theta}{T}\right)}{\left[\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]^2}-4\frac{\left(\frac{\alpha}{\nu^*}\right)^2}{\left[\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]^3}\right\}\right]$$

$$\nu^*\int_{-\infty}^{\infty} P_\nu\, d\left(\frac{\nu}{\nu^*}\right) = \frac{\alpha N_0\varepsilon^2}{3\mu c^2\nu^*}\left(\frac{\theta}{T}\right)\left(1-e^{-\theta/T}\right)\left[\left\{1+\gamma\left(2+\frac{1}{3}\frac{\theta}{T}\right)\right\}\frac{\pi}{\left(\frac{\alpha}{\nu^*}\right)}\right]+\cancel{3\frac{\pi}{2}\frac{1}{\left(\frac{\alpha}{\nu^*}\right)^3}}-\frac{3}{2}\pi\frac{1}{\left(\frac{\alpha}{\nu^*}\right)^3}\Big]$$

Overlapping Case

Emissivity of Diatomic Gases

$$P_\nu\, pL = \left(\frac{\alpha N_T \varepsilon^2 pL}{3\mu c^2 Q_{vjm}\, \nu^{*2}}\right) \sum_{j=1}^{\infty} \left[\frac{j(1-2\gamma j)\, e^{-\gamma\left(\frac{\theta}{T}\right) j(j+1)}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)+2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2} + \frac{j(1+2\gamma j)\, e^{-\gamma\left(\frac{\theta}{T}\right) j(j-1)}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)-2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2} \right]$$

the sum $= \displaystyle\int_0^{\infty} \left[\frac{j(1-2\gamma j)\, e^{-\gamma\left(\frac{\theta}{T}\right) j^2 - \gamma\left(\frac{\theta}{T}\right) j}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)+2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2} + \frac{j(1+2\gamma j)\, e^{-\gamma\left(\frac{\theta}{T}\right) j^2 + \gamma\left(\frac{\theta}{T}\right) j}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)-2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2} \right] dj$

$$-\frac{1}{6}\,\frac{1}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}$$

the integral $\cong \displaystyle\int_0^{\infty} \frac{(1-2\gamma j)\, e^{-\gamma\left(\frac{\theta}{T}\right) j}\left[\left\{\left(\frac{\nu}{\nu^*}-1\right)-2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2\right] + (1+2\gamma j)\, e^{+\gamma\left(\frac{\theta}{T}\right) j}\left[\left\{\left(\frac{\nu}{\nu^*}-1\right)+2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]}{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}^2 + 8\left\{\left(\frac{\alpha}{\nu^*}\right)^2-\left(\frac{\nu}{\nu^*}-1\right)^2\right\}\gamma^2 j^2 + 16\gamma^4 j^4}\, e^{-\gamma\left(\frac{\theta}{T}\right) j^2}\, j\, dj$

$$\cong \frac{T}{\gamma\theta}\int_0^{\infty} \frac{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\} + \left[4 + 4\left(\frac{\nu}{\nu^*}-1\right)\left(2+\frac{\theta}{T}\right) + \frac{\theta}{T}\left(2+\frac{1}{2}\frac{\theta}{T}\right)\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}\right]\gamma\frac{T}{\theta}\eta}{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}^2 + 8\left\{\left(\frac{\alpha}{\nu^*}\right)^2-\left(\frac{\nu}{\nu^*}-1\right)^2\right\}\left(\gamma\frac{T}{\theta}\right)\eta}\; e^{-\eta}\, d\eta$$

$$= \frac{T}{\gamma\theta}\,\frac{1}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}\int_0^{\infty}\left[1+\left\{\frac{4+4\left(\frac{\nu}{\nu^*}-1\right)\left(2+\frac{\theta}{T}\right)}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2} + \frac{\theta}{T}\left(2+\frac{1}{2}\frac{\theta}{T}\right) - \frac{8\left\{\left(\frac{\alpha}{\nu^*}\right)^2-\left(\frac{\nu}{\nu^*}-1\right)^2\right\}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}^2}\right\}\left(\gamma\frac{T}{\theta}\right)\eta\right] e^{-\eta}\, d\eta$$

$$= \frac{T}{\gamma\theta}\,\frac{1}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}\left[1+\gamma\frac{T}{\theta}\left\{\frac{4\left(\frac{\nu}{\nu^*}-1\right)\left(2+\frac{\theta}{T}\right)+4}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2} + \frac{\theta}{T}\left(2+\frac{1}{2}\frac{\theta}{T}\right) + 8\,\frac{\left(\frac{\nu}{\nu^*}-1\right)^2-\left(\frac{\alpha}{\nu^*}\right)^2}{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}^2}\right\}\right]$$

$$\boxed{\gamma\frac{T}{\theta} \ll \left(\frac{\alpha}{\nu^*}\right)^2}$$

$$\xi = \frac{\nu}{\nu^*}$$

$$\boxed{S_{\mathrm{sum}} \cong \frac{\frac{T}{\gamma\theta} + \left(2\frac{T}{\theta} + \frac{1}{3}\right)}{\xi^2 + \left(\frac{\alpha}{\nu^*}\right)^2} + 4\left(\frac{T}{\theta}\right)^2 \left[\frac{3 + \left(2 + \frac{\theta}{T}\right)\xi}{\left\{\xi^2 + \left(\frac{\alpha}{\nu^*}\right)^2\right\}^2} - \frac{4\left(\frac{\alpha}{\nu^*}\right)^2}{\left\{\xi^2 + \left(\frac{\alpha}{\nu^*}\right)^2\right\}^3} \right]}$$

Or roughly,

$$S_{\mathrm{sum}} \approx \frac{\frac{T}{\gamma\theta}}{\xi^2 + \left(\frac{\alpha}{\nu^*}\right)^2}$$

$$P_\nu \rho L = \frac{\alpha N_T \varepsilon^2 \rho L}{3\mu c^2 \nu^{*2}} \frac{1 - e^{-\theta/T}}{\cancel{\frac{T}{\gamma\theta}}} \frac{\cancel{\frac{T}{\gamma\theta}}}{\xi^2 + \left(\frac{\alpha}{\nu^*}\right)^2}$$

$$\varepsilon \cong \frac{15}{\pi^4} \frac{\left(\frac{\theta}{T}\right)^4}{\left(e^{\frac{\theta}{T}} - 1\right)} 4\sqrt{\frac{\alpha N_T \varepsilon^2 \rho L}{C\, 3\mu c^2 \nu^{*2}}} \left(1 - e^{-\theta/T}\right)^{1/2}$$

$$\boxed{\varepsilon \cong \frac{60}{\pi^4} \frac{\left(\frac{\theta}{T}\right)^4}{e^{\frac{1}{2}\frac{\theta}{T}} \left(e^{\frac{\theta}{T}} - 1\right)^{1/2}} \sqrt{\frac{N_T \varepsilon^2 \rho L}{3\mu c^2 \nu^*}} \left(\frac{\alpha}{C\nu^*}\right)^{1/2}}$$

$$\xi^2 = \left(\frac{\alpha}{\nu^*}\right)^2 \tan^2\varphi,$$

$$\frac{1}{\left(\frac{\alpha}{\nu^*}\right)^2} \cos^2\varphi$$

$$P_\nu \rho L = \frac{N_T \varepsilon^2 \rho L}{3\mu c^2 \alpha} \left[\left\{\frac{T}{\gamma\theta} + \left(2\frac{T}{\theta} + \frac{1}{3}\right)\right\} \cos^2\varphi + 4\left(\frac{T}{\theta}\right)^2 \left(\frac{\nu^*}{\alpha}\right)^2 \left\{ \left[3 + \left(\frac{\alpha}{\nu^*}\right)\left(2 + \frac{\theta}{T}\right)\tan\varphi\right] \cos^4\varphi - 4\cos^6\varphi \right\} \right]$$

Overlapping Case

<u>Emissivity of Diatomic Gases</u>

$$T_\nu pL = \left(\frac{\alpha N_T \varepsilon^2 pL}{3\mu c^2 Q_{\text{rot}}\, \nu^{*2}}\right) \sum_{j=1}^{\infty} \frac{j(1-2\gamma j)e^{-\gamma\left(\frac{B}{T}\right)j(j+1)}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)+2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2} + \frac{j(1+2\gamma j)e^{-\gamma\left(\frac{B}{T}\right)j(j-1)}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)-2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2}$$

The sum $=$

$$\int_0^\infty \Bigg[\frac{j(1-2\gamma j)(\cosh\gamma\frac{B}{T}j - \sinh\gamma\frac{B}{T}j)\,e^{-\gamma\left(\frac{B}{T}\right)j^2}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)+2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2}$$

$$+ \frac{j(1+2\gamma j)(\cosh\gamma\frac{B}{T}j + \sinh\gamma\frac{B}{T}j)\,e^{-\gamma\left(\frac{B}{T}\right)j^2}}{\left\{\left(\frac{\nu}{\nu^*}-1\right)-2\gamma j\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2}\Bigg]dj - \frac{1}{6}\,\frac{1}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}$$

$$= \int_0^\infty \frac{\left[\cosh\gamma\frac{B}{T}j + 2\gamma j\sinh\gamma\frac{B}{T}j\right]\left[\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}+4\gamma^2 j^2\right] + 4\left[\sinh\gamma\frac{B}{T}j + 2\gamma j\cosh\gamma\frac{B}{T}j\right]\left(\frac{\nu}{\nu^*}-1\right)\gamma j}{\left\{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2\right\}^2 - 8\left\{\left(\frac{\nu}{\nu^*}-1\right)^2-\left(\frac{\alpha}{\nu^*}\right)^2\right\}\gamma^2 j^2 + 16\gamma^4 j^4}\, e^{-\gamma\frac{B}{T}j^2}\, 2j\,dj$$

$$- \frac{1}{6}\,\frac{1}{\left(\frac{\nu}{\nu^*}-1\right)^2+\left(\frac{\alpha}{\nu^*}\right)^2}$$

Let $\underline{\left(\frac{\nu}{\nu^*}-1\right) = \left(\frac{\alpha}{\nu^*}\right)\xi}$ $\qquad 4\gamma^2\left(\frac{\nu^*}{\alpha}\right)^2 j^2 = x.$

$$\text{The sum} \cong \frac{1}{4\gamma^2}\int_0^\infty \frac{(\xi^2+1)+\left\{1+\frac{1}{4}\frac{\alpha^2}{\nu^{*2}}\left(\frac{1}{2}\frac{B^2}{T^2}+2\frac{B}{T}\right)(\xi^2+1)+4\frac{\alpha}{\nu^*}\left(\frac{B}{T}+2\right)\xi\right\}x}{\left\{(\xi^2-1)-x\right\}^2+4\xi^2}\, e^{-\frac{\left(\frac{\alpha}{\nu^*}\right)^2}{4\gamma}\frac{B}{T}x}\,dx$$

$$-\frac{1}{6}\left(\frac{\nu^*}{\alpha}\right)^2\frac{1}{\xi^2+1}$$

$$\frac{1}{\{x-(\xi^2-1)\}^2+4\xi^2} = \frac{1}{4i\xi}\left\{\frac{1}{x-(\xi^2-1)-2i\xi}-\frac{1}{x-(\xi^2-1)+2i\xi}\right\}$$

therefore the integral

$$= \frac{1}{4\gamma^2}\frac{1}{4i\xi}\left[(\xi^2+1)\int_0^\infty \cdots e^{-\frac{\left(\frac{\alpha}{\nu^*}\right)^2}{4\gamma}\left(\frac{\beta}{T}\right)x}\,dx\left\{\frac{1}{\cdot\ -2i\xi}-\frac{1}{\ +2i\xi}\right\}\right.$$

$$\left.+\left\{1+4\left(\frac{\beta}{T}+2\right)\frac{\alpha}{\nu^*}\xi+\frac{1}{4}\left(\frac{1}{2}\frac{\beta^2}{T^2}+2\frac{\beta}{T}\right)\frac{\alpha^2}{\nu^*}(\xi^2+1)\right\}\int_0^\infty e^{\cdots}\left\{\frac{x}{-2i\xi}-\frac{x}{+2i\xi}\right\}dx\right]$$

$$= \frac{1}{8\gamma^2\xi}\,\mathrm{Im}\left[\left[(\xi^2+1)+\{(\xi^2-1)+2i\xi\}\left\{1+4\left(\frac{\beta}{T}+2\right)\frac{\alpha}{\nu^*}\xi+\frac{1}{4}\left(\frac{1}{2}\frac{\beta^2}{T^2}+2\frac{\beta}{T}\right)\frac{\alpha^2}{\nu^*}(\xi^2+1)\right\}\right]\int_0^\infty\frac{e^{-\frac{\frac{\alpha^2}{\nu^*}}{4\gamma}\left(\frac{\beta}{T}\right)x}\,dx}{x-\{(\xi^2-1)+2i\xi\}}\right]$$

$$\cancel{= \frac{1}{8\gamma^2\xi}\,\mathrm{Im}\left[\{(\xi^2+1)+[(\xi^2-1)+2i\xi]\}\left[1+4\left(\frac{\beta}{T}+2\right)\frac{\alpha}{\nu^*}\xi+\frac{1}{4}\left(\frac{1}{2}\frac{\beta^2}{T^2}+2\frac{\beta}{T}\right)\frac{\alpha^2}{\nu^{*2}}(\xi^2+1)\right]\right\}}$$

$$\cancel{\cdot(-1)\,e^{-\frac{\frac{\alpha^2}{\nu^{*2}}}{4\gamma}\left(\frac{\beta}{T}\right)\{(\xi^2-1)+2i\xi\}}\,E_i\left(\frac{\frac{\alpha^2}{\nu^{*2}}}{4\gamma}\left(\frac{\beta}{T}\right)\{(\xi^2-1)+2i\xi\}\right)}$$

$$\cong \frac{1}{8\gamma^2\xi}\,\mathrm{Im}\left[-\left\{[(\xi^2+1)+[(\xi^2-1)+2i\xi]\right]\left[1+4\left(\frac{\beta}{T}+2\right)\frac{\alpha}{\nu^*}\xi+\frac{1}{4}\left(\frac{1}{2}\frac{\beta^2}{T^2}+2\frac{\beta}{T}\right)\frac{\alpha^2}{\nu^{*2}}(\xi^2+1)\right\}\left[2\log_\gamma(1+i\xi)\right]\right]$$

non-overlapping case

Emissivity of Diatomic Gases

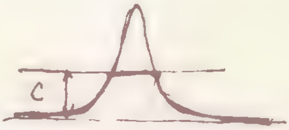
$$\boxed{\varepsilon \cong \frac{15}{\pi^4}\,\frac{\left(\frac{\theta}{T}\right)^4}{\left(e^{\frac{\theta}{T}}-1\right)}\int_{-\infty}^{\infty}\left(1-e^{-P_\nu pL}\right)d\left(\frac{\nu}{\nu^*}\right)}$$

$$P_\nu pL = \left(\frac{\alpha N_T \mathcal{E}^2 pL}{3\mu c^2 Q_{vjm}\nu^{*2}}\right)\sum_{j=1}^{\infty}\left[\frac{j(1-2\gamma j)\,e^{-\gamma\left(\frac{\theta}{T}\right)j(j+1)}}{\left\{\frac{\nu}{\nu^*}-(1-2\gamma j)\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2} + \frac{j(1+2\gamma j)\,e^{-\gamma\left(\frac{\theta}{T}\right)j(j-1)}}{\left\{\frac{\nu}{\nu^*}-(1+2\gamma j)\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2}\right]$$

$$\boxed{\int_{-\infty}^{\infty}P_\nu pL\,d\left(\frac{\nu}{\nu^*}\right) = \frac{\pi N_T \mathcal{E}^2 pL}{3\mu c^2 Q_{vjm}\nu^*}\sum_{j=1}^{\infty}\left[j(1-2\gamma j)\,e^{-\gamma\left(\frac{\theta}{T}\right)j(j+1)} + j(1+2\gamma j)\,e^{-\gamma\left(\frac{\theta}{T}\right)j(j-1)}\right]}$$

To chop off the peaks.

$$\frac{f(\eta)}{\eta^2+\varepsilon^2} = C$$



$$\eta = \pm\sqrt{\frac{f(\eta)}{C}-\varepsilon^2}$$

$$\int_{-\sqrt{}}^{+\sqrt{}}\left\{\frac{f(\eta)}{\eta^2+\varepsilon^2}-C\right\}d\eta = 2\int_0^{\sqrt{}}\left\{\frac{f(\eta)}{\eta^2+\varepsilon^2}-C\right\}d\eta$$

$$= 2\left[\frac{f(\eta)}{\varepsilon}\tan^{-1}\left(\frac{\sqrt{\frac{f(\eta)}{C}-\varepsilon^2}}{\varepsilon}\right) - C\sqrt{\frac{f(\eta)}{C}-\varepsilon^2}\right]$$

$$= 2\left[\frac{f(n)}{\varepsilon}\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{\varepsilon}{\sqrt{\frac{f(n)}{c}-\varepsilon^2}}\right)\right\} - c\sqrt{\frac{f(n)}{c}-\varepsilon^2}\right]$$

$$= \frac{\pi f(n)}{\varepsilon} - 2\left\{\frac{f(n)}{\varepsilon}\tan^{-1}\frac{\varepsilon}{\sqrt{\frac{f(n)}{c}-\varepsilon^2}} + c\sqrt{\frac{f(n)}{c}-\varepsilon^2}\right\}$$

$$\cong \frac{\pi f(n)}{\varepsilon} - 4c\sqrt{\frac{f(n)}{c}}$$

$$\therefore \boxed{\int_{-\infty}^{\infty}\left(1-e^{-P_\nu pL}\right)d\left(\frac{\nu}{\nu^0}\right) \cong 4\sqrt{\frac{\alpha N_r \bar{\varepsilon}^2 pL}{3\mu c^2 Q_{njm}\nu^{*2}C}} \sum_{j=1}^{\infty}\left[\sqrt{j(1-2\gamma j)}\, e^{-\frac{\chi}{2}\left(\frac{h}{T}\right)j(j+1)} + \sqrt{j(1+2\gamma j)}\, e^{-\frac{\chi}{2}\left(\frac{h}{T}\right)j(j-1)}\right]}$$

$$\frac{\partial}{\partial j}\sqrt{j(1-2\gamma j)}\, e^{-\frac{\chi}{2}\left(\frac{h}{T}\right)j(j+1)} = \left[\frac{1}{2}\frac{1-4\gamma j}{\sqrt{j(1-2\gamma j)}} - \sqrt{j(1-2\gamma j)}\,\frac{\chi}{2}\left(\frac{h}{T}\right)(2j+1)\right] e^{-\frac{\chi}{2}\frac{h}{T}j(j+1)}$$

So $$\sum_{j=1}^{\infty}\sqrt{j(1-2\gamma j)}\, e^{-\frac{\chi}{2}\left(\frac{h}{T}\right)j(j+1)} \cong \int_1^{\frac{1}{2\gamma}}\sqrt{j(1-2\gamma j)}\, e^{-\frac{\chi}{2}\left(\frac{h}{T}\right)j(j+1)}\,dj$$

$$+ \frac{1}{2} - \frac{1}{24}$$

$$\sum_{j=1}^{\infty}\sqrt{j(1+2\gamma j)}\, e^{-\frac{\chi}{2}\left(\frac{h}{T}\right)j(j-1)} \cong \int_1^{\infty}\sqrt{j(1+2\gamma j)}\, e^{-\frac{\chi}{2}\left(\frac{h}{T}\right)j(j-1)}\,dj + \frac{1}{2} - \frac{1}{24}$$

$$\boxed{\varepsilon \cong \frac{60}{\pi^4}\,\frac{\left(\frac{\theta}{T}\right)^{4.5}}{e^{\frac{3}{2}\left(\frac{\theta}{T}\right)}\left(e^{\theta/T}-1\right)^{1/2}}\left(\frac{\gamma\alpha}{c\nu^*}\right)^{\frac{1}{2}}\left(\frac{N_L\varepsilon^2 pL}{3\mu c^2\nu^*}\right)^{\frac{1}{2}} S\left(\gamma\frac{\theta}{T}\right)}$$

$$\boxed{S\left(\gamma\frac{\theta}{T}\right) = \sum_{j=1}^{j=\frac{1}{2\gamma}} \sqrt{j(1-2\gamma j)}\; e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j(j+1)} + \sum_{j=1}^{\infty} \sqrt{j(1+2\gamma j)}\; e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j(j-1)}}$$

Now let $\sqrt{1-2\gamma j} \cong e^{-\gamma j}$

$$\sum_{j=1}^{j=\frac{1}{2\gamma}} \sqrt{j(1-2\gamma j)}\; e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j(j+1)} = \frac{11}{24} + \int_1^\infty \sqrt{j}\; e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)\left\{j^2+j+2\left(\frac{T}{\theta}\right)j\right\}}\, dj$$

$$\cong -\frac{5}{24} + \int_0^\infty \sqrt{j}\; e^{-\gamma\left\{\frac{1}{2}\left(\frac{\theta}{T}\right)j^2 + \left(\frac{1}{2}\frac{\theta}{T}+1\right)j\right\}}\, dj$$

The integral $$I_1 = \int_0^\infty \sqrt{j}\; e^{-\gamma\left\{\frac{1}{2}\left(\frac{\theta}{T}\right)j^2 + \left(\frac{1}{2}\frac{\theta}{T}+1\right)j\right\}}\, dj$$

$$\int_0^\infty I_1\, e^{-\gamma t}\, d\gamma = \int_0^\infty \frac{\sqrt{j}\, dj}{\frac{1}{2}\left(\frac{\theta}{T}\right)j^2 + \left(\frac{1}{2}\frac{\theta}{T}+1\right)j + t}$$

$$= 2\left(\frac{T}{\theta}\right)\int_0^\infty \frac{\sqrt{j}\, dj}{j^2 + \left(1+2\frac{T}{\theta}\right)j + \left(2\frac{T}{\theta}t\right)} = 2\left(\frac{T}{\theta}\right)\int_0^\infty \frac{\sqrt{j}\, dj}{\left\{j+\left(\frac{1}{2}+\frac{T}{\theta}\right)\right\}^2 + \left\{2\frac{T}{\theta}t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2\right\}}$$

$$= 2\left(\frac{T}{\theta}\right)\left[\int_0^\infty \frac{\sqrt{j}\, dj}{\left\{j+\frac{1}{2}+\frac{T}{\theta} + i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}\right\}\left\{j+\frac{1}{2}+\frac{T}{\theta} - i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}\right\}}\right.$$

$$\int_0^\infty I_1\, e^{-\gamma t}\, d\gamma = \frac{\left(\frac{T}{\theta}\right)}{i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}} \int_0^\infty \left\{ \frac{1}{\xi + \frac{1}{2} + \frac{T}{\theta} - i\sqrt{\quad}} - \frac{1}{\xi + \frac{1}{2} + \frac{T}{\theta} + i\sqrt{\quad}} \right\} \sqrt{\xi}\, d\xi$$

$$= \frac{2\left(\frac{T}{\theta}\right)}{i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}} \int_0^\infty \left[\frac{\eta^2}{\eta^2 + \left\{\frac{1}{2} + \frac{T}{\theta} - i\sqrt{\quad}\right\}} - \frac{\eta^2}{\eta^2 + \left\{\frac{1}{2} + \frac{T}{\theta} + i\sqrt{\quad}\right\}} \right] d\eta$$

$$= \frac{2\left(\frac{T}{\theta}\right)}{i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}} \int_0^\infty \left[\frac{\left\{\frac{1}{2} + \frac{T}{\theta} + i\sqrt{\quad}\right\}}{\eta^2 + \left\{\frac{1}{2} + \frac{T}{\theta} + i\sqrt{\quad}\right\}} - \frac{\left\{\frac{1}{2} + \frac{T}{\theta} - i\sqrt{\quad}\right\}}{\eta^2 + \left\{\frac{1}{2} + \frac{T}{\theta} - i\sqrt{\quad}\right\}} \right] d\eta$$

$$= \frac{\pi\left(\frac{T}{\theta}\right)}{i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}} \left[\sqrt{\frac{1}{2} + \frac{T}{\theta} + i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}} - \sqrt{\frac{1}{2} + \frac{T}{\theta} - i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}} \right]$$

$$= \frac{2\pi\left(\frac{T}{\theta}\right)\left(2\left(\frac{T}{\theta}\right)t\right)^{1/4}}{\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}} \sqrt{\frac{1 - \dfrac{\frac{1}{2}+\frac{T}{\theta}}{\sqrt{2\left(\frac{T}{\theta}\right)t}}}{2}}$$

$$= \frac{2\pi\left(\frac{T}{\theta}\right)}{\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}} \frac{1}{\sqrt{2}} \sqrt{\sqrt{2\left(\frac{T}{\theta}\right)t} - \left(\frac{1}{2}+\frac{T}{\theta}\right)}$$

$$= \sqrt{2}\,\pi\left(\frac{T}{\theta}\right) \frac{1}{\left\{\sqrt{2\left(\frac{T}{\theta}\right)t} + \left(\frac{1}{2}+\frac{T}{\theta}\right)\right\}^{1/2}}$$

$$= \frac{\sqrt{2}\,\pi\left(\frac{T}{\theta}\right)}{\left(2\frac{T}{\theta}t\right)^{1/4}} \left\{ 1 + \frac{\frac{1}{2}+\frac{T}{\theta}}{\sqrt{2\frac{T}{\theta}}} \frac{1}{t^{1/2}} \right\}^{-1/2}$$

$$= \frac{\sqrt{2}\,\pi\left(\frac{T}{\theta}\right)}{\left(2\frac{T}{\theta}t\right)^{1/4}} \left\{ 1 - \frac{1}{2} \frac{\frac{1}{2}+\frac{T}{\theta}}{\sqrt{2\frac{T}{\theta}}} \frac{1}{t^{1/2}} + \frac{3}{8} \frac{\left(\frac{1}{2}+\frac{T}{\theta}\right)^2}{2\frac{T}{\theta}} \frac{1}{t} \cdots \right\}$$

$$\int_0^\infty I_1 e^{-\gamma t} d\gamma = \frac{\sqrt{2}\,\pi\left(\frac{T}{\theta}\right)}{\left(2\frac{T}{\theta}\right)^{1/4}} \left\{ \frac{1}{t^{1/4}} - \frac{1}{2}\frac{\frac{1}{2}+\frac{T}{\theta}}{\sqrt{2\frac{T}{\theta}}}\frac{1}{t^{3/4}} + \frac{3}{8}\frac{\left(\frac{1}{2}+\frac{T}{\theta}\right)^2}{\left(2\frac{T}{\theta}\right)}\frac{1}{t^{5/4}} \cdots \right\}$$

$$\therefore\ I_1 = \frac{\sqrt{2}\,\pi\left(\frac{T}{\theta}\right)}{\left(2\frac{T}{\theta}\right)^{1/4}} \left\{ \frac{1}{\Gamma\left(\frac{1}{4}\right)}\frac{1}{\gamma^{3/4}} - \frac{1}{2}\frac{\frac{1}{2}+\frac{T}{\theta}}{\sqrt{2\frac{T}{\theta}}}\frac{1}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\gamma^{1/4}} + \frac{3}{8}\frac{\left(\frac{1}{2}+\frac{T}{\theta}\right)^2}{2\frac{T}{\theta}}\frac{1}{\Gamma\left(\frac{5}{4}\right)}\gamma^{1/4} \cdots \right\}$$

$$= \pi\sqrt[4]{2}\left(\frac{T}{\gamma\theta}\right)^{3/4}\frac{1}{\Gamma\left(\frac{1}{4}\right)}\left\{1 - \frac{1}{2}\frac{\frac{1}{2}+\frac{T}{\theta}}{\sqrt{2\frac{T}{\gamma\theta}}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} + \frac{3}{2}\frac{\left(\frac{1}{2}+\frac{T}{\theta}\right)^2}{2\frac{T}{\gamma\theta}} + \cdots\cdots \right\}$$

$$\boxed{I_1 = \pi\sqrt[4]{2}\left(\frac{T}{\gamma\theta}\right)^{3/4}\frac{1}{\Gamma\left(\frac{1}{4}\right)}\left\{1 - \frac{1}{4}\frac{\Gamma^2\left(\frac{1}{4}\right)}{\pi}\left(\frac{1}{2}+\frac{T}{\theta}\right)\sqrt{\frac{\gamma\theta}{T}} + \frac{3}{4}\left(\frac{1}{2}+\frac{T}{\theta}\right)^2\left(\frac{\gamma\theta}{T}\right) + \cdots \right\}}$$

Similarly
$$\sum_{j=1}^{\infty}\sqrt{j(1+2\gamma j)}\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j(j-1)} = -\frac{5}{24} + \underbrace{\int_0^\infty \sqrt{j}\, e^{-\gamma\left\{\frac{1}{2}\left(\frac{\theta}{T}\right)j^2 - \frac{1}{2}\left(\frac{\theta}{T}\right)j - j\right\}} dj}_{I_2}$$

So
$$\int_0^\infty e^{-\gamma t} I_2\, d\gamma = \int_0^\infty \frac{\sqrt{j}\, dj}{\frac{1}{2}\left(\frac{\theta}{T}\right)j^2 - \left(\frac{1}{2}\frac{\theta}{T}+1\right)j + t}$$

$$= 2\left(\frac{T}{\theta}\right)\int_0^\infty \frac{\sqrt{j}\, dj}{j^2 - \left(1+2\frac{T}{\theta}\right)j + 2\left(\frac{T}{\theta}\right)t}$$

$$= \frac{2\left(\frac{T}{\theta}\right)}{i\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}}\int_0^\infty \left[\frac{\frac{1}{2}+\frac{T}{\theta}+i\sqrt{}}{\eta^2 - \left\{\frac{1}{2}+\frac{T}{\theta}+i\sqrt{}\right\}} - \frac{\frac{1}{2}+\frac{T}{\theta}-i\sqrt{}}{\eta^2 - \left\{\frac{1}{2}+\frac{T}{\theta}-i\sqrt{}\right\}}\right] d\eta$$

$$= \frac{\pi\left(\frac{T}{\theta}\right)}{\sqrt{2\left(\frac{T}{\theta}\right)t - \left(\frac{1}{2}+\frac{T}{\theta}\right)^2}}\left[\sqrt{\frac{1}{2}+\left(\frac{T}{\theta}\right)+i\sqrt{}} + \sqrt{\frac{1}{2}+\frac{T}{\theta}-i\sqrt{}}\right]$$

$$\int_0^{\infty} e^{-\gamma t} I_2 \, d\gamma = \frac{\sqrt{2}\,\pi\left(\frac{T}{\theta}\right)}{\left(2\frac{T}{\theta}t\right)^{1/4}} \left\{ 1 - \frac{\frac{1}{2}+\frac{T}{\theta}}{\sqrt{2\frac{T}{\theta}}} \frac{1}{t^{1/2}} \right\}^{-1/2}$$

$$\therefore \quad \boxed{S\left(\gamma, \frac{\theta}{T}\right) = -\frac{5}{12} + \frac{2\pi(2)^{1/4}}{\Gamma\left(\frac{1}{4}\right)} \left(\frac{T}{\gamma\theta}\right)^{3/4} \left\{ 1 + \frac{3}{16}\left(1+2\frac{T}{\theta}\right)^2 \left(\frac{\gamma\theta}{T}\right) + \cdots \right\}}$$

$$\boxed{\varepsilon \cong \frac{60}{\pi^4} \frac{\left(\frac{\theta}{T}\right)^5}{e^{\frac{1}{2}\left(\frac{\theta}{T}\right)}\left(e^{\theta/T}-1\right)^{1/2}} \left(\frac{\gamma\alpha}{c\nu^4}\right)^{\frac{1}{2}} \left(\frac{N_0 \varepsilon^2 \delta L}{3\mu c^2 \nu^4}\right)^{\frac{1}{2}} S\left(\gamma, \frac{\theta}{T}\right)} \qquad 1 \ \frac{\theta}{T}$$

z	$1-e^{-z}$
0	1
0.25	0.22120
0.50	0.39347
0.75	0.52763
1.00	0.63212
1.50	0.77687
2.00	0.86466
2.50	0.91792
3.00	0.95021
4.00	0.98168
5.00	0.99326
6.00	0.99752
7.00	0.99909

$C \sim 2.2$?

For CO at 300°K, $\alpha = 0.077\ \text{cm}^{-1}$

$\gamma = 0.000895$

$\theta = 3066.9$

$\theta/T = 10.223$, $\nu^* = 2142.3$

$\frac{I}{\gamma\theta} = 109.2$

$$\frac{N_I \varepsilon^2}{3\mu c^2 \nu^*} = \frac{246.904}{3.1416 \times 2142.3} = 0.03666 = 3.666 \times 10^{-2}$$

$$\frac{\gamma\alpha}{c\nu^*} = \frac{0.000895 \times 0.077}{2.2 \times 2142.3} = 10^{-8}\,\frac{8.95 \times 0.77}{2.2 \times 2.1423}$$

$$= 1.460 \times 10^{-8}$$

$$\begin{array}{r} 12.78 \\ 10.45 \\ \hline 2.33 \end{array}$$

$$\frac{\left(\frac{\theta}{T}\right)^{4.5}}{e^{\frac{3}{2}\left(\frac{\theta}{T}\right)}\left(e^{\theta/T}-1\right)^{1/2}} \sim \left(\frac{\theta}{T}\right)^{4.5} e^{-1.5\left(\frac{\theta}{T}\right)} = e^{10.45 - 10.22} = 1.259$$

$$\varepsilon \cong \underline{0.617}\overset{1.259}{\swarrow} \times 10^{-5} \times \sqrt{1.460 \times 3.666} \times S$$

$$= 1.797 \times 10^{-5}\, S$$

$\Gamma(\tfrac{1}{4}) = 3.6256$

$$S = -0.4165 + \frac{2 \times 3.1416 \times 1.189}{3.6256}\, 33.8 \left\{1 + \cdots\right\} = 69.7$$

$$\varepsilon \cong 1.253 \times 10^{-3} \sqrt{pL}$$

$$\boxed{\varepsilon = \frac{c_1}{\sigma T^4}\int_0^\infty \frac{\nu^5}{e^{c_2\nu/T}-1}\left\{1 - e^{-P_\nu\cdot(pL)}\right\}d\nu}$$

ε = emissivity
σ = Stephan-Boltzmann constant
T = absolute temperature °K
$c_1 = 2\pi c^2 h = 3.732 \times 10^{-5}$ erg.-cm^2 sec^{-1}
$c_2 = ch/k = 1.432$ cm-°K.
pL = optical density

$$\boxed{P_\nu \cong \frac{\alpha}{\pi}\sum_{j=1}^{\infty}\left[\frac{S^{0\to1}_{j\to j-1}}{(\nu-\nu^{0\to1}_{j\to j-1})^2+\alpha^2} + \frac{S^{0\to1}_{j-1\to j}}{(\nu-\nu^{0\to1}_{j-1\to j})^2+\alpha^2}\right]}$$

α = spectral half-width

$$S^{0\to1}_{j\to j-1} \cong \left(\frac{N_T \pi \mathcal{E}^2}{3\mu c^2 Q_{vjm}}\right)\frac{\nu^{0\to1}_{j\to j-1}\, j\, e^{-\frac{E_{0,j}}{kT}}}{\nu^{0\to1}_{0\to0}}$$

$$S^{0\to1}_{j-1\to j} \cong \left(\frac{N_T \pi \mathcal{E}^2}{3\mu c^2 Q_{vjm}}\right)\frac{\nu^{0\to1}_{j-1\to j}\, j\, e^{-\frac{E_{0,j-1}}{kT}}}{\nu^{0\to1}_{0\to0}}$$

$$\nu^{0\to1}_{j\to j-1} = \left|\frac{E_{0j} - E_{1,j-1}}{hc}\right| , \quad \nu^{0\to1}_{j-1\to j} = \left|\frac{E_{0,j-1} - E_{1,j}}{hc}\right|$$

$$\nu^{0\to1}_{0\to0} = \left|\frac{E_{0,0} - E_{1,0}}{hc}\right|$$

For the sake of simplicity,

$$\boxed{\frac{E_{n,j}}{hc} \cong \left(\frac{kT}{hc}\right)\left[nu - xn(n-1)u + j(j+1)\sigma'\right]}$$

where $u = \frac{\theta}{T}$, $\sigma' = \frac{B_v hc}{kT} = \frac{B_e hc}{kT}\left(1 - \frac{\delta}{2}\right)$

$$\frac{E_{0,j}}{kT} = j(j+1)\sigma'$$

$$\frac{E_{0,j-1}}{kT} = (j-1)j\,\sigma'$$

$$\boxed{\nu^{0\to1}_{j\to j-1} = \left(\frac{kT}{hc}\right)\left[u - 2\sigma' j\right] \quad ; \quad \nu^{0\to1}_{j-1\to j} = \left(\frac{kT}{hc}\right)\left[u + 2\sigma' j\right]}$$

$$\nu^{n\to1}_{n\to 0} = \left(\frac{kT}{hc}\right)u$$

So

$$\boxed{P_\nu = \left(\frac{\alpha N_T \varepsilon^2}{3\mu c^2 Q_{njm}}\right)\sum_{j=0}^{\infty}\frac{j\left[1 - 2\left(\frac{\sigma'}{u}\right)j\right]e^{-\sigma' j(j+1)}}{\left\{\left(\nu - \frac{kT}{hc}u + 2\frac{kT\sigma'}{hc}j\right\}^2 + \alpha^2} + \frac{j\left[1 + 2\left(\frac{\sigma'}{u}\right)j\right]e^{-\sigma' j(j-1)}}{\left\{\left(\nu - \frac{kT}{hc}u - 2\frac{kT\sigma'}{hc}j\right\}^2 + \alpha^2}}$$

$$P_\nu = \frac{\alpha N_T \varepsilon^2}{3\mu c^2 Q_{njm}}\left(\frac{hc}{ukT}\right)^2\left[\sum_{j=0}^{\infty}\frac{j\left[1 - 2\left(\frac{\sigma'}{u}\right)j\right]e^{-\sigma' j(j+1)}}{\left\{\left(\frac{\nu hc}{ukT}\right) - 1 + 2\left(\frac{\sigma'}{u}\right)j\right\}^2 + \left(\frac{\alpha hc}{ukT}\right)^2} + \sum_{j=0}^{\infty}\frac{j\left[1 + 2\frac{\sigma'}{u}j\right]e^{-\sigma' j(j-1)}}{\left\{\left(\frac{\nu hc}{ukT}\right) - 1 - 2\left(\frac{\sigma'}{u}\right)j\right\}^2 + \left(\frac{\alpha hc}{ukT}\right)^2}\right]$$

$$ukT = \frac{\theta}{T}kT = k\theta.$$

$$\frac{\sigma'}{u} = \frac{B_v hc}{k\theta} = \beta\,, \qquad \frac{\alpha hc}{ukT} = \frac{\alpha hc}{k\theta} = \gamma.$$

Or

$$P_\nu = \frac{\alpha N_T \mathcal{E}^2}{3\mu c^2 Q_{njm}}\left(\frac{hc}{k\theta}\right)^2\left[\sum_{j=0}^{\infty}{}' \frac{j(1-2\beta j)e^{-\sigma' j(j+1)}}{\left\{\left(\frac{\nu hc}{k\theta}\right)-1+2\beta j\right\}^2+\gamma^2} + \sum_{j=0}^{\infty}{}' \frac{j(1+2\beta j)e^{-\sigma' j(j-1)}}{\left\{\left(\frac{\nu hc}{k\theta}\right)-1-2\beta j\right\}^2+\gamma^2}\right]$$

$$\cong \frac{\alpha N_T \mathcal{E}^2}{3\mu c^2 Q_{njm}}\left(\frac{hc}{k\theta}\right)^2\left[\int_0^\infty \frac{j(1-2\beta j)e^{-\sigma' j(j+1)}}{\left\{\left(\frac{\nu hc}{k\theta}\right)-1+2\beta j\right\}^2+\gamma^2}\,dj + \int_0^\infty \frac{j(1+2\beta j)e^{-\sigma' j(j-1)}}{\left\{\left(\frac{\nu hc}{k\theta}\right)-1-2\beta j\right\}^2+\gamma^2}\,dj\right.$$

$$\left. - \frac{1}{6}\,\frac{1}{\left(\frac{\nu hc}{k\theta}-1\right)^2+\gamma^2}\right]$$

Now $\qquad \sigma' T = O(1), \quad$ so $\quad \sigma' \ll 1.$

$$\beta = \frac{\sigma' T}{nT} = \frac{\sigma' T}{\theta} \ll 1.$$

$$\frac{\nu hc}{k\theta} = \frac{\nu hc}{hc\omega^*} = \frac{\nu}{\omega^*}$$

$$\frac{hc}{k\theta} = \frac{1}{\omega^{*2}}$$

$$\gamma = \frac{\alpha}{\omega^*} \ll 1$$

$$\frac{\alpha N_T \mathcal{E}^2}{3\mu c^2 Q_{njm}}\left(\frac{hc}{k\theta}\right)^2 = \frac{\alpha N_T \mathcal{E}^2}{3\mu c^2 Q_{njm}\,\omega^{*2}}$$

$$\boxed{P_\nu \rho L = \frac{\alpha N_T \varepsilon^2 \rho L}{3\mu c^2 Q_{mjm} \omega^{*2}} \left[\int_0^\infty \frac{j(1-2\beta j) e^{-\sigma' j(j+1)}}{\left\{\frac{\nu}{\omega^*} - 1 + 2\beta j\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} dj + \int_0^\infty \frac{j(1+2\beta j) e^{-\sigma' j(j-1)}}{\left\{\frac{\nu}{\omega^*} - 1 - 2\beta j\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} dj - \frac{1}{6} \frac{1}{\left(\frac{\nu}{\omega^*} - 1\right)^2 + \left(\frac{\alpha}{\omega^*}\right)^2} \right]}$$

Let $j = t - \frac{1}{2}$ in the first integral and
$j = t + \frac{1}{2}$ in the second integral

The sum of integrals is

$$I = e^{\frac{\sigma'}{4}} \left\{ \int_{\frac{1}{2}}^\infty \frac{(t-\frac{1}{2})(1+\beta-2\beta t) e^{-\sigma' t^2} dt}{\left\{\left(\frac{\nu}{\omega^*} - 1 - \beta\right) + 2\beta t\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} + \int_{\frac{1}{2}}^\infty \frac{(t+\frac{1}{2})(1+\beta+2\beta t) e^{-\sigma' t^2} dt}{\left\{\left(\frac{\nu}{\omega^*} - 1 - \beta\right) - 2\beta t\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} \right.$$

$$\left. + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(t+\frac{1}{2})(1+\beta+2\beta t) e^{-\sigma' t^2} dt}{\left\{\left(\frac{\nu}{\omega^*} - 1 - \beta\right) - 2\beta t\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} \right\}$$

But

$$\frac{(t-\frac{1}{2})(1+\beta-2\beta t)}{\left\{\left(\frac{\nu}{\omega^*} - 1 - \beta\right) + 2\beta t\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} - \frac{(-t-\frac{1}{2})(1+\beta+2\beta t)}{\left\{\left(\frac{\nu}{\omega^*} - 1 - \beta\right) - 2\beta t\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2}$$

$$= \frac{\left\{\left(\frac{\nu}{\omega^*} - 1 - \beta\right)\left[(1+2\beta)\frac{\nu}{\omega^*} - (1+\beta)\right] + \left(\frac{\alpha}{\omega^*}\right)^2 (1+2\beta)\right\} + 8\beta^2\left(\frac{\nu}{\omega^*} - \frac{1}{2}\right)t^2}{\left\{\left(\frac{\nu}{\omega^*} - 1 - \beta\right)^2 - 4\beta^2 t^2\right\}^2 + 2\left(\frac{\alpha}{\omega^*}\right)^2 \left\{\left(\frac{\nu}{\omega^*} - 1 - \beta\right)^2 + 4\beta^2 t^2\right\} + \left(\frac{\alpha}{\omega^*}\right)^4} \, 2t$$

Putting $\quad \sigma'(z - \frac{1}{4}) = \xi \quad , \qquad \beta^2 z = \frac{\beta^2}{\sigma'}\xi + \frac{1}{4}\beta^2$

t^2

$t + \frac{1}{2} = \eta$

$$I = \frac{1}{\sigma'} \int_0^{\infty} \frac{\left[\left\{ \left(\frac{\nu}{\omega^*} - 1 - \beta\right)\left(\frac{\nu}{\omega^*} - 1\right) + \left(\frac{\alpha}{\omega^*}\right)^2 + 2\left(\frac{\nu}{\omega^*} - \frac{1}{2}\right)\beta^2 \right\} + 8\frac{\beta^2}{\sigma'}\left(\frac{\nu}{\omega^*} - \frac{1}{2}\right)\xi \right] e^{-\xi}\, d\xi}{\left\{ \left(\frac{\nu}{\omega^*} - 1 - \beta\right)^2 - \beta^2 - 4\frac{\beta^2}{\sigma'}\xi \right\}^2 + 2\left(\frac{\alpha}{\omega^*}\right)^2 \left\{ \left(\frac{\nu}{\omega^*} - 1 - \beta\right)^2 + \beta^2 + 4\frac{\beta^2}{\sigma'}\xi \right\} + \left(\frac{\alpha}{\omega^*}\right)^4}$$

$$+ \int_0^1 \frac{\eta\, d\eta}{\left\{ \left(\frac{\nu}{\omega^*} - 1\right) - 2\beta\eta \right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2}$$

$$\boxed{\begin{aligned} I = \frac{1}{\beta^2} \int_0^{\infty} & \frac{8\left(\frac{\nu}{\omega^*} - \frac{1}{2}\right)\xi + \left\{ \frac{\sigma'}{\beta^2}\left(\frac{\nu}{\omega^*} - 1 - \beta\right)\left(\frac{\nu}{\omega^*} - 1\right) + \left(\frac{\alpha}{\omega^*}\right)^2 \frac{\sigma'}{\beta^2} + 2\sigma'\left(\frac{\nu}{\omega^*} - \frac{1}{2}\right) \right\} e^{-\xi}\, d\xi}{\left\{ (4\xi + \sigma') - \frac{\sigma'}{\beta^2}\left(\frac{\nu}{\omega^*} - 1 - \beta\right)^2 \right\}^2 + 2\frac{\sigma'}{\beta^2}\left(\frac{\alpha}{\omega^*}\right)^2 \left\{ (4\xi + \sigma') + \frac{\sigma'}{\beta^2}\left(\frac{\nu}{\omega^*} - 1 - \beta\right)^2 \right\} + \left(\frac{\sigma'}{\beta^2}\frac{\alpha^2}{\omega^{*2}}\right)^2} \\ & + \int_0^1 \frac{\eta\, d\eta}{\left\{ \left(\frac{\nu}{\omega^*} - 1\right) - 2\beta\eta \right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} \end{aligned}}$$

For non-overlapping case

$$P_\nu \rho L = \left(\frac{\alpha N_T \varepsilon^2 \rho L}{3\mu c^2 Q_{nijm}\,\omega^{*2}}\right) \sum_{j=1}^{\infty} \left\{ \frac{j(1-2\beta j)\, e^{-\sigma' j(j+1)}}{\left\{\frac{\nu}{\omega^*} - (1-2\beta j)\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} + \frac{j(1+2\beta j)\, e^{-\sigma' j(j-1)}}{\left\{\frac{\nu}{\omega^*} - (1+2\beta j)\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2} \right\}$$

$$\downarrow \xi$$

$$\int_0^\infty J_\nu \, d\nu = \sum_{j=1}^{\infty} J^0\left(\nu^{0\to 1}_{j\to j-1}\right) \int_{-\infty}^{\infty} \left(1 - e^{-\xi \frac{j(1-2\beta j)e^{-\sigma' j(j+1)}}{\left\{\frac{\nu}{\omega^*} - (1-2\beta j)\right\}^2 + \left(\frac{\alpha}{\omega^*}\right)^2}}\right) d\nu$$

$$+ \; J^0\left(\nu^{0\to 1}_{j-1\to j}\right) \int_{-\infty}^{\infty} \left(1 - \qquad \right) d\nu$$

$$= \cancel{2\pi\alpha} \sum_{j=1}^{\infty} J^0\left(\nu^{0\to 1}_{j\to j-1}\right) x_j e^{-x_j} \left\{ I_0(x_j) + I_1(x_j) \right\}$$

$$+ \; J^0\left(\nu^{0\to 1}_{j-1\to j}\right) x_{j-1} e^{-x_{j-1}} \left\{ I_0(x_{j-1}) \right\}$$

$$\omega^* \int_{-\infty}^{\infty} \left(1 - e^{\cancel{-\xi j(1-2\beta j) e^{-\sigma' j(j+1)} \frac{1}{\eta^2 + \left(\frac{\alpha}{\omega^*}\right)^2}}} \right) d\eta$$

$$= 2\pi\alpha \sum_{j=1}^{\infty} \Big[J^0\left(\nu^{0\to 1}_{j\to j-1}\right) x_j e^{-x_j} \left\{ I_0(x_j) + I_1(x_j) \right\}$$

$$+ \; J^0\left(\nu^{0\to 1}_{j-1\to j}\right) x_{j-1} e^{-x_{j-1}} \left\{ I_0(x_{j-1}) + I_1(x_{j-1}) \right\} \Big]$$

$$x_j = \frac{1}{2}\xi\left(\frac{\omega^*}{\alpha}\right)^2 j(1-2\beta j)\, e^{-\sigma' j(j+1)} \; ; \quad x_{j-1} = \frac{1}{2}\xi\left(\frac{\omega^*}{\alpha}\right)^2 j(1+2\beta j)\, e^{-\sigma' j(j-1)}$$

Emissivity of Diatomic Gases

I. Formulation

$$\boxed{\varepsilon = \frac{\displaystyle\int_0^\infty \frac{\nu^3\{1-e^{-P_\nu pL}\}\,d\nu}{e^{c_2\nu/T}-1}}{\displaystyle\int_0^\infty \frac{\nu^3\,d\nu}{e^{c_2\nu/T}-1}}}$$

where $\qquad c_2 = ch/k$

$$P_\nu = \frac{\alpha}{\pi}\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=1}^{\infty}\left[\frac{S_{j\to j-1}^{m\to m+l}}{\left(\nu-\nu_{j\to j-1}^{m\to m+l}\right)^2+\alpha^2}+\frac{S_{j-1\to j}^{m\to m+l}}{\left(\nu-\nu_{j-1\to j}^{m\to m+l}\right)^2+\alpha^2}\right]$$

$$S_{j\to j+1}^{m\to m+l} = \frac{N_T\pi\varepsilon^2}{3\mu c^2 Q_{njm}}\,m\,\frac{\nu_{j\to j-1}^{m\to m+l}}{\nu_{0\to 0}^{0\to 1}}\,j\,e^{-\frac{E_{m,j}}{kT}}\left[1+4\varepsilon j\left(1+\frac{5\varepsilon j}{8}-\frac{3\varepsilon}{8}\right)\right]\cdot\left[1-e^{-\left(hc\nu_{j\to j-1}^{m\to m+l}/kT\right)}\right]$$

$m\to m+1$
$0\to 1$

$$S_{j-1\to j}^{m\to m+l} = \frac{N_T\pi\varepsilon^2}{3\mu c^2 Q_{njm}}\,\frac{\nu_{j-1\to j}^{m\to m+l}}{\nu_{0\to 0}^{m\to m+l}}\,j\,e^{-\frac{E_{m,j-1}}{kT}}\left[1-4\varepsilon j\left(1-\frac{5\varepsilon j}{8}-\frac{3\varepsilon}{8}\right)\right]\left[1-e^{-\left(hc\nu_{j-1\to j}^{m\to m+l}/kT\right)}\right]$$

$$\varepsilon = \frac{h}{4\pi^2 I c\nu_{0\to 0}^{m\to m+l}} = \frac{2k(\sigma T)}{ch\nu_{0\to 0}^{m\to m+l}}$$

$$\nu_{j\to j+1}^{m\to m+l} = \frac{1}{hc}\left| E_{m,j} - E_{m+l,j+1} \right|$$

$$\nu_{j-1\to j}^{m\to m+l} = \frac{1}{hc}\left| E_{m,j-1} - E_{m+l,j} \right|$$

$$\nu_{0\to 0}^{m\to m+l} = \frac{1}{hc}\left| E_{m,0} - E_{m+l,0} \right|$$

$$E_{m,j} = (kT)\left[u\cdot m - xu\, m(m-1) + j(j+1)6\left\{ 1 - 4\gamma^2 j(j+1) - \delta m \right\} \right]$$

$$= (k\theta)\left[m - \underline{x}\cdot m(m-1) + \gamma\, j(j+1)\left\{ 1 - 4\gamma^2 j(j+1) - \underline{\delta} m \right\} \right]$$

$$\gamma = \frac{B_0}{\nu^*}$$

$$\delta \cong 6\gamma\left[\left(\frac{x}{\gamma}\right)^{\frac{1}{2}} - 1 \right]$$

$$\varepsilon = \frac{2\gamma}{l\left\{ 1 - x(l+2m-1) \right\}}$$

$$c_2\nu/T = \frac{ch}{k}\frac{\nu}{T} = \frac{ch\nu^*}{kT}\frac{\nu}{\nu^*} = \left(\frac{\theta}{T}\right)\frac{\nu}{\nu^*}$$

$\frac{\alpha}{\nu^*}$,

$$\frac{\nu_{j\to j-1}^{m\to m+l}}{\nu^*} = \frac{k\theta}{hc\nu^*}(\qquad) = 1(\qquad)$$

$$\frac{\nu_{j \to j-1}^{m \to m+l}}{\nu_{0 \to 0}^{m \to m+l}} = \frac{l\{1 - x(l+2m-1)\} - \gamma j\{2 + 16\gamma^2 j^2 + \delta[(j-1)l - 2jm]\}}{l\{1 - x(l+2m-1)\}}$$

$$\frac{E_{m,j}}{kT} = \left(\frac{b}{T}\right)\left[m - x(m-1)m + \gamma j(j+1)\{1 - 4\gamma^2 j(j+1) - \delta m\}\right]$$

$$\frac{E_{m,j-1}}{kT} = \left(\frac{b}{T}\right)\left[m - x(m-1)m + \gamma(j-1)j\{1 - 4\gamma^2(j-1)j - \delta m\}\right]$$

$$P_\nu \, p L = \left(\frac{\alpha N_T \,\mathcal{E}^2 \, p L}{3\mu c^2 Q_{njm} \, \nu^{*2}}\right) \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} \sum_{l=1}^{\infty} \left[\frac{\left[1-\gamma \dfrac{j\{2-16\gamma^2 j^2+\delta[(j-1)l-2m]\}}{l\{1-x(l+2m-1)\}}\right] j \, e^{-\left(\frac{h}{T}\right)\left[m-x(m-1)m+\gamma j(j+1)\{1-4\gamma^2 j(j+1)-\delta m\}\right]}}{\left\{\frac{\nu}{\nu^*}-\left[l\{1-x(l+2m-1)\}-\gamma j\{2-16\gamma^2 j^2+\delta[(j-1)l-2m]\}\right]\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2} \cdot F.G \right.$$

$$\left. + \frac{\left[1+\gamma \dfrac{j\{2-16\gamma^2 j^2-\delta[(j+1)l+2m]\}}{l\{1-x(l+2m-1)\}}\right] j \, e^{-\frac{h}{T}\left[m-x(m-1)m+\gamma j(j-1)\{1-4\gamma^2 j(j-1)-\delta m\}\right]}}{\left\{\frac{\nu}{\nu^*}-\left[l\{1-x(l+2m-1)\}+\gamma j\{2-16\gamma^2 j^2-\delta[(j+1)l+2m]\}\right]\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2} \cdot F'G' \right]$$

where

$$F = 1 + 4\varepsilon j\left(1+\frac{5\varepsilon j}{8}-\frac{3\varepsilon}{8}\right), \quad F' = 1 - 4\varepsilon j\left(1-\frac{5\varepsilon j}{8}-\frac{3\varepsilon}{8}\right)$$

$$G = 1 - \exp\left\{-\left(\frac{h}{T}\right)\left[l\{1-x(l+2m-1)\}-\gamma j\{2-16\gamma^2 j^2+\delta[(j-1)l-2m]\}\right]\right\}$$

$$G' = 1 - \exp\left\{-\left(\frac{h}{T}\right)\left[l\{1-x(l+2m-1)\}+\gamma j\{2-16\gamma^2 j^2-\delta[(j+1)l+2m]\}\right]\right\}$$

$$\varepsilon = \frac{2\gamma}{l\{1-x(l+2m-1)\}}$$

$$Q_{njm} = \frac{1}{\sigma(1-e^{-\theta/T})}\left[1+\frac{\sigma}{3}+\frac{8\gamma^2}{\sigma}+\frac{\delta}{e^{\theta/T}-1}+\frac{2x\cdot\frac{\theta}{T}}{(e^{\theta/T}-1)^2}\right]$$

$$\sigma = \gamma\left(\frac{\theta}{T}\right)$$

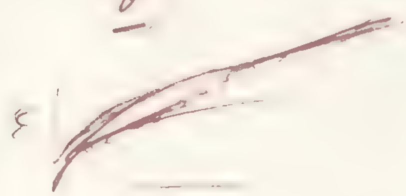
$$\boxed{Q_{njm} = \frac{(T/\theta)}{\gamma(1-e^{-\theta/T})}\left[1+\frac{\gamma}{3}\left(\frac{\theta}{T}\right)+8\gamma\left(\frac{T}{\theta}\right)+\frac{\delta}{e^{\theta/T}-1}+\frac{2x\cdot\frac{\theta}{T}}{(e^{\theta/T}-1)^2}\right]}$$

$$P_\nu pL = \frac{N_T\,\varepsilon^2 pL}{\mu c^2 \alpha}$$

$$\boxed{\delta} \qquad \frac{B_0}{\nu^*} = \frac{\sigma}{u} = \frac{\sigma T}{\theta}$$

$$\boxed{X}$$

$$\frac{T}{\theta}$$



$\varepsilon =$

Quadratic Approximation

Emissivity of Diatomic Gases

I. Formulation

$$\boxed{\varepsilon = \frac{\displaystyle\int_0^\infty \frac{\nu^3\left\{1 - e^{-P_\nu p L}\right\} d\nu}{e^{\left(\frac{a}{T}\right)\frac{\nu}{\nu^*}} - 1}}{\displaystyle\int_0^\infty \frac{\nu^3\, d\nu}{e^{\frac{a}{T}\frac{\nu}{\nu^*}} - 1}}}$$

$$P_\nu \cong \frac{\alpha}{\pi} \sum_{j=1}^{\infty} \left[\frac{S^{0\to1}_{j\to j-1}}{\left(\nu - \nu^{0\to1}_{j\to j-1}\right)^2 + \alpha^2} + \frac{S^{0\to1}_{j-1\to j}}{\left(\nu - \nu^{0\to1}_{j-1\to j}\right)^2 + \alpha^2} \right]$$

$$S^{0\to1}_{j\to j-1} = \frac{N_T \varepsilon^2 \pi}{3\mu c^2 Q_{njm}} \frac{\nu^{0\to1}_{j\to j-1}}{\nu^{0\to1}_{0\to0}}\, j\, e^{-\frac{E_{0,j}}{kT}}\, F\cdot G$$

$$S^{0\to1}_{j-1\to j} = \frac{N_T \varepsilon^2 \pi}{3\mu c^2 Q_{njm}} \frac{\nu^{0\to1}_{j-1\to j}}{\nu^{0\to1}_{0\to0}}\, j\, e^{-\frac{E_{0,j-1}}{kT}}\, F'\cdot G'$$

$$\left.\begin{aligned} F &= 1 + 4\lambda j\left(1 + \frac{5\lambda j}{8} - \frac{3\lambda}{8}\right) \\ F' &= 1 - 4\lambda j\left(1 - \frac{5\lambda j}{8} - \frac{3\lambda}{8}\right) \end{aligned}\right\} \quad \lambda = \frac{2k(\sigma T)}{ch\,\nu^{0\to1}_{0\to0}}$$

$$G = 1 - \exp\left\{-\left(\frac{hc}{kT}\right)\nu^{0\to1}_{j\to j-1}\right\}$$

$$G' = 1 - \exp\left\{-\left(\frac{hc}{kT}\right)\nu^{0\to1}_{j-1\to j}\right\}$$

$$\nu^{0\to 1}_{j\to j-1} = \frac{1}{hc}\left| E_{0,j} - E_{1,j-1} \right|$$

$$\nu^{0\to 1}_{j-1\to j} = \frac{1}{hc}\left| E_{0,j-1} - E_{1,j} \right|$$

$$\nu^{0\to 1}_{0\to 0} = \frac{1}{hc}\left| E_{0,0} - E_{1,0} \right|$$

$$\boxed{E_{n,j} = (kE)\left[n - x\,n(n-1) + \gamma\, j(j+1)\left\{ 1 - 4\gamma^2 j(j+1) - \delta n \right\} \right]}$$

where x, γ, δ are all small and non-dimensional

$$\boxed{\gamma = \frac{B_0}{\nu^*} \quad ; \quad \delta \cong 6\gamma\left[\left(\frac{x}{\gamma}\right)^{1/2} - 1 \right]}$$

Then $\boxed{\lambda = 2\gamma}$ Approx.

$$\boxed{P_\nu p L = \left(\frac{\alpha N_T \varepsilon^2 p L}{3\mu c^2 Q_{njm} \nu^{*2}} \right) \sum_{j=1}^{\infty} \left[\frac{j(1-2\gamma j)\, e^{-\left(\frac{\theta}{T}\right)\gamma j(j+1)}}{\left\{ \frac{\nu}{\nu^*} - (1-2\gamma j) \right\}^2 + \left(\frac{\alpha}{\nu^*}\right)^2} + \frac{j(1+2\gamma j)\, e^{-\left(\frac{\theta}{T}\right)\gamma j(j-1)}}{\left\{ \frac{\nu}{\nu^*} - (1+2\gamma j) \right\}^2 + \left(\frac{\alpha}{\nu^*}\right)^2} \right]}$$

$$\boxed{Q_{njm} = \frac{(T/\theta)}{\gamma\left(1 - e^{-\theta/T}\right)} \left[1 + \gamma\left\{ \frac{1}{3}\left(\frac{\theta}{T}\right) + 8\left(\frac{T}{\theta}\right) \right\} + \frac{\delta}{e^{(\theta/T)} - 1} + \frac{2x\left(\frac{\theta}{T}\right)}{\left\{ e^{(\theta/T)} - 1 \right\}^2} \right]}$$

$$\varepsilon = \frac{\displaystyle\int_0^\infty \frac{\left(\frac{\nu}{\nu^*}\right)^3 \left\{ 1 - e^{-P_\nu p L} \right\} d\left(\frac{\nu}{\nu^*}\right)}{e^{\left(\frac{\theta}{T}\right)\frac{\nu}{\nu^*}} - 1}}{\displaystyle\int_0^\infty \frac{\left(\frac{\nu}{\nu^*}\right)^3 d\left(\frac{\nu}{\nu^*}\right)}{e^{(\theta/T)\frac{\nu}{\nu^*}} - 1}}$$

First
$$\int_0^\infty \frac{\eta^3\, d\eta}{e^{\left(\frac{\theta}{T}\right)\eta}-1} = \left(\frac{T}{\theta}\right)^4 \int_0^\infty \frac{\xi^3\, d\xi}{e^{\xi}-1}$$

<u>then approximately</u>,

$$\boxed{\varepsilon \cong \frac{\left(\frac{\theta}{T}\right)^4}{\left(e^{\frac{\theta}{T}}-1\right)\int_0^\infty \frac{\xi^3\, d\xi}{e^{\xi}-1}} \int_{-\infty}^{\infty} \left(1- e^{-P_\nu pL}\right) d\left(\frac{\nu}{\nu^*}\right)}$$

When the pL is very small, $1- e^{-P_\nu pL} \cong P_\nu pL$.

So
$$\varepsilon \cong \frac{(\theta/T)^4}{\left(e^{\frac{\theta}{T}}-1\right)\int_0^\infty \frac{\xi^3\, d\xi}{e^{\xi}-1}} \int_{-\infty}^{\infty} (P_\nu pL)\, d\left(\frac{\nu}{\nu^*}\right)$$

But
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = 2\pi i \frac{1}{2ia} = \frac{\pi}{a}$$

$\therefore$

$$\boxed{\int_{-\infty}^{\infty} (P_\nu pL)\, d\left(\frac{\nu}{\nu^*}\right) = \frac{\pi N_T \varepsilon^2 pL}{3\mu c^2 Q_{mjm} \nu^*} {\sum_{j=1}^{\infty}}' \Big[j(1-2\gamma j)\, e^{-\left(\frac{\theta}{T}\right)\gamma j(j+1)} + j(1+2\gamma j)\, e^{-\left(\frac{\theta}{T}\right)\gamma j(j-1)} \Big]}$$

Now $\displaystyle\sum_{j=0}^{\infty}{}' \left[j(1-2\gamma j)\, e^{-\left(\frac{\theta}{T}\right)\gamma j(j+1)} + j(1+2\gamma j)\, e^{-\left(\frac{\theta}{T}\right)\gamma j(j-1)} \right]$

$$\cong \int_0^{\infty} j(1-2\gamma j)\, e^{-\left(\frac{\theta}{T}\right)\gamma j(j+1)}\, dj + \int_0^{\infty} j(1+2\gamma j)\, e^{-\left(\frac{\theta}{T}\right)\gamma j(j-1)}\, dj$$

$$- \frac{1}{6}$$

$$= -\frac{1}{6} + \int_{\frac{1}{2}}^{\infty} (t-\tfrac{1}{2})\left\{1-2\gamma(t-\tfrac{1}{2})\right\} e^{-\gamma\left(\frac{\theta}{T}\right)(t^2-\frac{1}{4})}\, dt$$

$$+ \int_{-\frac{1}{2}}^{\infty} (t+\tfrac{1}{2})\left\{1+2\gamma(t+\tfrac{1}{2})\right\} e^{-\gamma\left(\frac{\theta}{T}\right)(t^2-\frac{1}{4})}\, dt$$

$$= -\frac{1}{6} + \int_{\frac{1}{2}}^{\infty} \left[(t-\tfrac{1}{2})\left\{(1+\gamma)-2\gamma t\right\} + (t+\tfrac{1}{2})\left\{(1+\gamma)+2\gamma t\right\} \right] e^{-\gamma\left(\frac{\theta}{T}\right)(t^2-\frac{1}{4})}\, dt$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} (t+\tfrac{1}{2})\left\{(1+\gamma)+2\gamma t\right\} e^{-\gamma\left(\frac{\theta}{T}\right)(t^2-\frac{1}{4})}\, dt$$

$$\cong -\frac{1}{6} + \int_0^1 \eta\, d\eta + \int_{\frac{1}{2}}^{\infty} 2(1+\gamma)t\; e^{-\gamma\left(\frac{\theta}{T}\right)(t^2-\frac{1}{4})}\, dt$$

$$= -\frac{1}{6} + \frac{1}{2} + \int_0^{\infty} e^{-\gamma\left(\frac{\theta}{T}\right)z}\, dz = -\frac{1}{6} + \frac{1}{2} + \frac{1}{\gamma\left(\frac{\theta}{T}\right)}$$

$$= \frac{1}{3} + \frac{T/\theta}{\gamma}$$

$$\boxed{K = \int_0^\infty \frac{\xi^3\, d\xi}{e^\xi - 1}}$$

Therefore

$$\boxed{\varepsilon \cong \frac{\gamma(\theta/T)^5\left(\frac{1}{3} + \frac{T/\theta}{\gamma}\right)e^{-(\theta/T)}}{K\left[1+\gamma\left\{\frac{1}{3}\frac{\theta}{T} + 8\frac{T}{\theta}\right\}\right]}\left(\frac{\pi N_T \varepsilon^2 \rho L}{3\mu c^2 \nu^*}\right)}$$

$$K = \int_0^\infty \xi^3 \sum_{n=1}^\infty e^{-n\xi}\, d\xi = \sum_{n=1}^\infty \int_0^\infty \xi^3 e^{-n\xi}\, d\xi$$

$$= \sum_{n=1}^\infty \frac{1}{n^4}\int_0^\infty e^{-t}\, t^{4-1}\, dt = 6\sum_{n=1}^\infty \frac{1}{n^4} = \frac{6\pi^4}{90}$$

$$= \frac{\pi^4}{15}$$

$$\boxed{\varepsilon \cong \frac{15}{\pi^4}\; \frac{\gamma\left(\frac{\theta}{T}\right)^5\left(\frac{1}{3} + \frac{T/\theta}{\gamma}\right)e^{-\left(\frac{\theta}{T}\right)}}{1+\gamma\left\{\frac{1}{3}\frac{\theta}{T} + 8\frac{T}{\theta}\right\}}\left(\frac{\pi N_T\, \varepsilon^2}{3\mu c^2}\,\frac{\rho L}{\nu^*}\right)}$$

For CO $\quad \gamma = \frac{1.916}{2142.3} = 0.000895 \quad ; \quad \nu^* = 2142.3$

$\theta = 3066.9°$ $\quad$ 2.324

Let $\quad T = 300°, \quad \frac{\theta}{T} = 10.223$ $\quad$ 11.62 / 10.22 / 1.40

$$\left(\frac{\theta}{T}\right)^5 e^{-\frac{\theta}{T}} = (10.223)^5\, e^{-10.223} = e^{11.62 - 10.22} = 4.06$$

$$\frac{\pi N_T \varepsilon^2}{3\mu c^2 \nu^*} = \frac{246.904}{2142.3} = 0.1152$$

$$\frac{15}{\pi^4}\; 0.000895 \times 4.06 \times (0.3333 + 109.2) \times 0.1152 = 0.00706$$

Now if we replace $1 - e^{-P_\nu pL}$ by

$$a\left(P_\nu pL\right) - b\left(P_\nu pL\right)^2$$

then
$$\int_{-\infty}^{\infty}\left(1 - e^{-P_\nu pL}\right) d\left(\frac{\nu}{\nu^*}\right) = a\int_{-\infty}^{\infty}\left(P_\nu pL\right) d\left(\frac{\nu}{\nu^*}\right) - b\int_{-\infty}^{\infty}\left(P_\nu pL\right)^2 d\left(\frac{\nu}{\nu^*}\right)$$

$$= a\left(\frac{\pi N_\gamma \varepsilon^2 pL}{3\mu c^2 Q_{mjm}\nu^*}\right)\left(\frac{1}{3} + \frac{T/k}{\gamma}\right)$$

$$- b\left(\frac{\alpha N_\gamma \varepsilon^2 pL}{3\mu c^2 Q_{mjm}\nu^{*2}}\right)^2 \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\left[j(1-2\gamma j)m(1-2\gamma m)e^{-\gamma\left(\frac{h}{T}\right)\left\{j(j+1)+m(m+1)\right\}}\right.$$

$$\int_{-\infty}^{\infty}\frac{d\left(\frac{\nu}{\nu^*}\right)}{\left[\left\{\frac{\nu}{\nu^*}-(1-2\gamma j)\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]\left[\left\{\frac{\nu}{\nu^*}-(1-2\gamma m)\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]}$$

$$+ 2j(1+2\gamma j)m(1-2\gamma m)e^{-\gamma\left(\frac{h}{T}\right)\left\{j(j-1)+m(m+1)\right\}}\int_{-\infty}^{\infty}\frac{d\left(\frac{\nu}{\nu^*}\right)}{\left[\left\{\frac{\nu}{\nu^*}-(1+2\gamma j)\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]\left[\left\{\frac{\nu}{\nu^*}-(1-2\gamma m)\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]}$$

$$\left. + j(1+2\gamma j)m(1+2\gamma m)e^{-\gamma\left(\frac{h}{T}\right)\left\{j(j-1)+m(m-1)\right\}}\int_{-\infty}^{\infty}\frac{d\left(\frac{\nu}{\nu^*}\right)}{\left[\left\{\frac{\nu}{\nu^*}-(1+2\gamma j)\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]\left[\left\{\frac{\nu}{\nu^*}-(1+2\gamma m)\right\}^2+\left(\frac{\alpha}{\nu^*}\right)^2\right]}\right]$$

But $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{[(x-x_1)^2+a^2][(x-x_2)^2+a^2]} = \int_{-\infty}^{\infty} \frac{dx}{[x-(x_1+ia)][x-(x_1-ia)][x-(x_2+ia)][x-(x_2-ia)]}$

$$= 2\pi i \left\{ \frac{1}{2ia\{(x_1-x_2)^2+2ia(x_1-x_2)\}} + \frac{1}{2ia\{(x_1-x_2)^2-2ia(x_1-x_2)\}} \right\}$$

$$= \frac{\pi}{a}\,\frac{1}{(x_1-x_2)}\;\frac{2(x_1-x_2)}{(x_1-x_2)^2+4a^2}$$

$$= 2\frac{\pi}{a}\,\frac{1}{(x_1-x_2)^2+4a^2}$$

$$\therefore \int_{-\infty}^{\infty} (pL\cdot P_\nu)^2\, d\left(\frac{\nu}{\nu^*}\right) = \frac{\pi}{2}\left(\frac{\nu^*}{\alpha}\right)\left(\frac{\alpha N_T \varepsilon^2 pL}{3\mu c^2 Q_{njm}\, \nu^{*2}}\right)^2 \sum_{j=1}^{\infty}{}' \sum_{m=1}^{\infty}{}' \Bigg[$$

$$\frac{j(1-2\gamma j)\,m(1-2\gamma m)\,e^{-\gamma\left(\frac{h}{T}\right)\{j(j+1)+m(m+1)\}}}{\gamma^2(j-m)^2+\left(\frac{\alpha}{\nu^*}\right)^2} + 2\,\frac{j(1+2\gamma j)\,m(1-2\gamma m)\,e^{-\gamma\left(\frac{h}{T}\right)\{j(j-1)+m(m+1)\}}}{\gamma^2(j+m)^2+\left(\frac{\alpha}{\nu^*}\right)^2}$$

$$+ \frac{j(1+2\gamma j)\,m(1+2\gamma m)\,e^{-\gamma\left(\frac{h}{T}\right)\{j(j-1)+m(m-1)\}}}{\gamma^2(j-m)^2+\left(\frac{\alpha}{\nu^*}\right)^2}\Bigg]$$

Now
$$\sum_{j=0}^{\infty}{}'\sum_{m=0}^{\infty}{}' \frac{j(1-2\gamma j)m(1-2\gamma m)\,e^{-\gamma\left(\frac{h}{T}\right)\{j(j+1)+m(m+1)\}}}{1+\left(\gamma\frac{\nu^*}{\alpha}\right)^2(j-m)^2}$$

$$= \sum_{j=0}^{\infty}{}' j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{h}{T}\right)j(j+1)} + 2\sum_{s=1}^{\infty}{}' \frac{1}{1+\left(\gamma\frac{\nu^*}{\alpha}\right)^2 s^2}\sum_{j=s}^{\infty}{}' j(1-2\gamma j)(j-s)\{1-2\gamma(j-s)\}\, e^{-\gamma\left(\frac{h}{T}\right)\{j(j+1)+(j-s)(j-s+1)\}}$$

$$\sum_{j=0}^{\infty}{}'\sum_{m=0}^{\infty}{}' \frac{j(1+2\gamma j)m(1+2\gamma m)\,e^{-\gamma\left(\frac{h}{T}\right)\{j(j-1)+m(m-1)\}}}{1+\left(\gamma\frac{\nu^*}{\alpha}\right)^2(j-m)^2}$$

$$= \sum_{j=0}^{\infty}{}' j^2(1+2\gamma j)^2 e^{-2\gamma\left(\frac{h}{T}\right)j(j-1)} + 2\sum_{s=1}^{\infty}{}' \frac{1}{1+\left(\gamma\frac{\nu^*}{\alpha}\right)^2 s^2}\sum_{j=s}^{\infty}{}' j(1+2\gamma j)(j-s)\{1+2\gamma(j-s)\}\, e^{-\gamma\left(\frac{h}{T}\right)\{j(j-1)+(j-s)(j-s-1)\}}$$

$$2\sum_{j=1}^{\infty}{}'\sum_{m=1}^{\infty}{}' \frac{j(1+2\gamma j)m(1-2\gamma m)\,e^{-\gamma\left(\frac{h}{T}\right)\{j(j-1)+m(m+1)\}}}{1+\left(\gamma\frac{\nu^*}{\alpha}\right)^2(j+m)^2}$$

$$= 2\sum_{s=1}^{\infty}{}' \frac{1}{1+\left(\gamma\frac{\nu^*}{\alpha}\right)^2 s^2}\sum_{j=0}^{j=s}{}' j(1+2\gamma j)(s-j)\{1-2(s-j)\}\, e^{-\gamma\left(\frac{h}{T}\right)\{j(j-1)+(s-j)(s-j+1)\}}$$

$$\gamma\frac{\nu^*}{\alpha} = \frac{B_0}{\alpha}$$

Therefore

$$\int_{-\infty}^{\infty} (pL\cdot P_\nu)^2 \, d\left(\frac{\nu}{\nu^*}\right) = \frac{1}{2\pi}\left(\frac{\nu^*}{\alpha}\right)\left(\frac{\pi N_T \varepsilon^2 pL}{3\mu c^2 Q_{vjm}\nu^*}\right)^2 \Bigg[\sum_{j=1}^{\infty} \Big\{ j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{B}{T}\right)j(j+1)} +$$

$$+ j^2(1+2\gamma j)^2 e^{-2\gamma\left(\frac{B}{T}\right)j(j-1)} \Big\}$$

$$+ 2\sum_{s=1}^{\infty} \frac{1}{1+\left(\frac{B_0}{\alpha}\right)^2 s^2} \Big\{ \sum_{j=s}^{\infty} j(1-2\gamma j)(j-s)\{1-2\gamma(j-s)\} e^{-\gamma\left(\frac{B}{T}\right)\{j(j+1)+(j-s)(j-s+1)\}}$$

$$+ \sum_{j=s}^{\infty} j(1+2\gamma j)(j-s)\{1+2\gamma(j-s)\} e^{-\gamma\left(\frac{B}{T}\right)\{j(j-1)+(j-s)(j-s-1)\}}$$

$$+ \sum_{j=0}^{j=s} j(1+2\gamma j)(s-j)\{1-2\gamma(s-j)\} e^{-\gamma\left(\frac{B}{T}\right)\{j(j-1)+(s-j)(s-j+1)\}} \Big\} \Bigg]$$

$$\frac{\partial}{\partial j}\left\{ j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{B}{T}\right)j(j+1)} \right\} = \left\{ 2j(1-2\gamma j)^2 - 2j^2(1-2\gamma j)2\gamma - j^2(1-2\gamma j)^2 2\gamma\left(\frac{B}{T}\right)(2j+1) \right\}$$

$$e^{-2\gamma\left(\frac{B}{T}\right)j(j+1)}$$

Therefore $$\sum_{j=1}^{\infty} j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{B}{T}\right)j(j+1)}$$

$$= \int_1^{\infty} j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{B}{T}\right)j(j+1)}\, dj + \frac{1}{2}(1-2\gamma)^2 e^{-4\gamma\left(\frac{B}{T}\right)}$$

$$- \frac{1}{12}\left\{ 2(1-2\gamma)^2 - 4\gamma(1-2\gamma) - (1-2\gamma)^2 2\gamma\left(\frac{B}{T}\right)3 \right\} e^{-4\gamma\left(\frac{B}{T}\right)}$$

$$= \int_1^{\infty} j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{B}{T}\right)j(j+1)}\, dj + \left\{ \frac{1}{3}(1-2\gamma)^2 + \frac{1}{3}\gamma(1-2\gamma) + \frac{1}{2}\gamma\left(\frac{B}{T}\right)(1-2\gamma)^2 \right\} e^{-4\gamma\left(\frac{B}{T}\right)}$$

$$\sum_{j=1}^{\infty} j^2(1+2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)j(j-1)}$$

$$= \int_1^{\infty} j^2(1+2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)j(j-1)}\,dj + \frac{1}{2}(1+2\gamma)^2$$

$$-\frac{1}{12}\left\{2(1+2\gamma)^2 + 4\gamma(1+2\gamma) - (1+2\gamma)^2 2\gamma\left(\frac{\theta}{T}\right)\cdot 1\right\}$$

$$= \int_1^{\infty} j^2(1+2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)j(j-1)}\,dj + \left\{\frac{1}{3}(1+2\gamma)^2 - \frac{1}{3}\gamma(1+2\gamma) + \frac{1}{6}\gamma\left(\frac{\theta}{T}\right)(1+2\gamma)^2\right\}$$

Now $\displaystyle\int_1^{\infty} j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)j(j+1)}\,dj + \int_1^{\infty} j^2(1+2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)j(j-1)}\,dj$

$$= \int_{\frac{3}{2}}^{\infty} \left(\eta-\tfrac{1}{2}\right)^2(1+\gamma-2\gamma\eta)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)\left(\eta^2-\frac{1}{4}\right)}\,d\eta + \int_{\frac{1}{2}}^{\infty} \left(\eta+\tfrac{1}{2}\right)^2(1+\gamma+2\gamma\eta)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)\left(\eta^2-\frac{1}{4}\right)}\,d\eta$$

A simpler approximate formula for $\left(\frac{\theta}{T}\right)$ not too large is

$$\sum_{j=1}^{\infty}\left\{j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)j(j+1)} + j^2(1+2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)j(j-1)}\right\}$$

$$\simeq \int_{-\infty}^{\infty} j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)j(j+1)}\,dj$$

$$= \int_{-\infty}^{\infty} \left(\eta-\tfrac{1}{2}\right)^2(1+\gamma-2\gamma\eta)^2 e^{-2\gamma\left(\frac{\theta}{T}\right)\left(\eta^2-\frac{1}{4}\right)}\,d\eta$$

$$= e^{\frac{\gamma}{2}\left(\frac{\theta}{T}\right)} \int_{-\infty}^{\infty} \left\{\eta^2 - \eta + \frac{1}{4}\right\}\left\{(1+\gamma)^2 - 4\gamma(1+\gamma)\eta + 4\gamma^2\eta^2\right\} e^{-2\gamma\left(\frac{\theta}{T}\right)\eta^2} d\eta$$

$$= 2 e^{\frac{\gamma}{2}\left(\frac{\theta}{T}\right)} \int_0^{\infty} \left[\frac{1}{4}(1+\gamma)^2 + \left\{\gamma^2 + 4\gamma(1+\gamma) + (1+\gamma)^2\right\}\eta^2 + 4\gamma^2\eta^4\right] e^{-2\gamma\left(\frac{\theta}{T}\right)\eta^2} d\eta$$

$$= e^{\frac{\gamma}{2}\left(\frac{\theta}{T}\right)} \int_0^{\infty} \left[\frac{\frac{1}{4}(1+\gamma)^2}{\sqrt{2\gamma\left(\frac{\theta}{T}\right)}} \zeta^{\frac{1}{2}-1} + \frac{\gamma^2 + 4\gamma(1+\gamma) + (1+\gamma)^2}{\left(2\gamma\frac{\theta}{T}\right)^{3/2}} \zeta^{\frac{3}{2}-1} + \frac{4\gamma^2}{\left(2\gamma\frac{\theta}{T}\right)^{5/2}} \zeta^{\frac{5}{2}-1}\right] e^{-\zeta} d\zeta$$

$$= \frac{\sqrt{\pi}\, e^{\frac{\gamma}{2}\left(\frac{\theta}{T}\right)}}{\sqrt{2\gamma\frac{\theta}{T}}} \left[\frac{1}{4}(1+\gamma)^2 + \frac{\gamma^2 + 4\gamma(1+\gamma) + (1+\gamma)^2}{2\gamma\left(\frac{\theta}{T}\right)} \frac{1}{2} + \frac{4\gamma^2}{\left(2\gamma\frac{\theta}{T}\right)^2} \frac{3}{4}\right]$$

$$= \frac{1}{4}\sqrt{\frac{\pi T}{2\gamma\theta}} \left[(1+\gamma)^2 + \left\{\gamma + 4(1+\gamma) + \frac{(1+\gamma)^2}{\gamma}\right\}\frac{T}{\theta} + 3\left(\frac{T}{\theta}\right)^2\right] e^{\frac{\gamma}{2}\left(\frac{\theta}{T}\right)} \sim 1$$

$$\boxed{\begin{aligned} &\sum_{j=1}^{\infty} \left\{ j^2(1-2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right) j(j+1)} + j^2(1+2\gamma j)^2 e^{-2\gamma\left(\frac{\theta}{T}\right) j(j-1)} \right\} \\ &\cong \frac{1}{4}\sqrt{\frac{\pi T}{2\gamma\theta}} \left[1 + \frac{1}{\gamma}\frac{T}{\theta} + 3\left(\frac{T}{\theta}\right)^2\right] \end{aligned}}$$

$$\sum_{j=s}^{\infty} j(1-2\gamma j)(j-s)\left\{1-2\gamma(j-s)\right\} e^{-\gamma\left(\frac{\theta}{T}\right)\left\{j(j+1) + (j-s)(j-s+1)\right\}}$$

$$\cong \int_s^{\infty} j(1-2\gamma j)(j-s)\left\{1-2\gamma(j-s)\right\} e^{-\gamma\left(\frac{\theta}{T}\right)\left\{j(j+1) + (j-s)(j-s+1)\right\}} dj - \frac{1}{12} s(1-2\gamma s) e^{-\gamma\left(\frac{\theta}{T}\right)s(s+1)}$$

$$= \int_s^{\infty} j(1-2\gamma j)(j-s)\left\{1-2\gamma(j-s)\right\} e^{-2\gamma\left(\frac{\theta}{T}\right)\left\{\left(j-\frac{s-1}{2}\right)^2 + \frac{s^2-1}{4}\right\}} dj - \frac{1}{12} s(1-2\gamma s) e^{-\gamma\frac{\theta}{T}s(s+1)}$$

Put $\qquad j-\frac{s-1}{2}=\eta\,, \qquad j=\eta+\frac{s-1}{2}\,, \qquad j-s=\eta-\frac{s+1}{2}$

$$= \int_{\frac{s+1}{2}}^{\infty} \left(\eta+\frac{s-1}{2}\right)\left\{1-(s-1)\gamma-2\gamma\eta\right\}\left(\eta-\frac{s+1}{2}\right)\left\{1+(s+1)\gamma-2\gamma\eta\right\} e^{-2\gamma\left(\frac{\hbar}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}}\,d\eta$$

$$-\frac{1}{12}s(1-2\gamma s)\,e^{-\gamma\left(\frac{\hbar}{T}\right)s(s+1)}$$

$$\sum_{j=s}^{\infty} j(1+2\gamma j)(j-s)\left\{1+2\gamma(j-s)\right\} e^{-\gamma\left(\frac{\hbar}{T}\right)\left\{j(j-1)+(j-s)(j-s-1)\right\}}$$

$$= \int_{s}^{\infty} j(1+2\gamma j)(j-s)\left\{1+2\gamma(j-s)\right\} e^{-2\gamma\left(\frac{\hbar}{T}\right)\left\{\left(j-\frac{s+1}{2}\right)^2+\frac{s^2-1}{4}\right\}}\,dj-\frac{1}{12}s(1+2\gamma s)\,e^{-\gamma\left(\frac{\hbar}{T}\right)s(s-1)}$$

$$= \int_{\frac{s-1}{2}}^{\infty} \left(\eta+\frac{s+1}{2}\right)\left\{1+(s+1)\gamma+2\gamma\eta\right\}\left(\eta-\frac{s-1}{2}\right)\left\{1-(s-1)\gamma+2\gamma\eta\right\} e^{-2\gamma\left(\frac{\hbar}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}}\,d\eta$$

$$-\frac{1}{12}s(1+2\gamma s)\,e^{-\gamma\left(\frac{\hbar}{T}\right)s(s-1)}$$

$$\sum_{j=0}^{j=s}{}' \; j(1+2\gamma j)(s-j)\left\{1-2\gamma(s-j)\right\} e^{-\gamma\left(\frac{\hbar}{T}\right)\left\{j(j-1)+(s-j)(s-j+1)\right\}}$$

$$= \int_{-\frac{s+1}{2}}^{\frac{s-1}{2}} \left(\eta+\frac{s+1}{2}\right)\left\{1+(s+1)\gamma+2\gamma\eta\right\}\left(\frac{s-1}{2}-\eta\right)\left\{1-(s-1)\gamma+2\gamma\eta\right\} e^{-2\gamma\left(\frac{\hbar}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}}\,d\eta$$

$$-\frac{1}{12}s(1-2\gamma s)\,e^{-\gamma\left(\frac{\hbar}{T}\right)s(s+1)}-\frac{1}{12}s(1+2\gamma s)\,e^{-\gamma\left(\frac{\hbar}{T}\right)s(s-1)}$$

$$S(s) = \sum_{j=s}^{\infty} j(1-2\gamma j)(j-s)\{1-2\gamma(j-s)\}\, e^{-\gamma\left(\frac{\hbar}{T}\right)\{j(j+1)+(j-s)(j-s+1)\}}$$

$$+ \sum_{j=s}^{\infty} j(1+2\gamma j)(j-s)\{1+2\gamma(j-s)\}\, e^{-\gamma\left(\frac{\hbar}{T}\right)\{j(j-1)+(j-s)(j-s-1)\}}$$

$$+ \sum_{j=0}^{j=s} j(1+2\gamma j)(s-j)\{1-2\gamma(s-j)\}\, e^{-\gamma\left(\frac{\hbar}{T}\right)\{j(j-1)+(s-j)(s-j+1)\}}$$

$$= -2\int_{-\left(\frac{s-1}{2}\right)}^{\frac{s-1}{2}} \left(\eta+\tfrac{s-1}{2}\right)\left(\eta-\tfrac{s+1}{2}\right)\{1-(s-1)\gamma-2\gamma\eta\}\{1+(s+1)\gamma-2\gamma\eta\}\, e^{-2\gamma\left(\frac{\hbar}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}}\,d\eta$$

$$+2\int_{\frac{s-1}{2}}^{\frac{s+1}{2}} \left(\eta+\tfrac{s-1}{2}\right)\left(\eta-\tfrac{s+1}{2}\right)\{1-(s-1)\gamma-2\gamma\eta\}\{1+(s+1)\gamma-2\gamma\eta\}\, e^{-2\gamma\left(\frac{\hbar}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}}\,d\eta$$

$$+\int_{-\infty}^{\infty} \left(\eta+\tfrac{s-1}{2}\right)\left(\eta-\tfrac{s+1}{2}\right)\{1-(s-1)\gamma-2\gamma\eta\}\{1+(s+1)\gamma-2\gamma\eta\}\, e^{-2\gamma\left(\frac{\hbar}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}}\,d\eta$$

$$-\frac{1}{6}s\left\{(1-2\gamma s)\,e^{-\gamma\left(\frac{\hbar}{T}\right)s(s+1)} + (1+2\gamma s)\,e^{-\gamma\left(\frac{\hbar}{T}\right)s(s-1)}\right\}$$

$$\int_{-\infty}^{\infty} \left(\eta+\tfrac{s-1}{2}\right)\left(\eta-\tfrac{s+1}{2}\right)\{1-(s-1)\gamma-2\gamma\eta\}\{1+(s+1)\gamma-2\gamma\eta\}\, e^{-2\gamma\left(\frac{\hbar}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}}\,d\eta$$

$$=2e^{\frac{\gamma}{2}\left(\frac{\hbar}{T}\right)(s^2-1)}\int_{0}^{\infty}\Big[4\gamma^2\eta^4+\big\{[1-(s-1)\gamma][1+(s+1)\gamma]+4\gamma(1+\gamma)-(s^2-1)\gamma^2\big\}\eta^2$$

$$-\frac{s^2-1}{4}[1-(s-1)\gamma][1+(s+1)\gamma]\Big]\,e^{-2\gamma\left(\frac{\hbar}{T}\right)\eta^2}\,d\eta$$

$$= \frac{\frac{1}{4}\sqrt{\left(\frac{\pi}{2\gamma}\right)\left(\frac{T}{\theta}\right)}\left[3\left(\frac{T}{\theta}\right)^2 + \frac{1+2\gamma+4\gamma(1+\gamma)-2(s^2-1)\gamma^2}{\gamma}\left(\frac{T}{\theta}\right) - (s^2-1)\left\{1+2\gamma-(s^2-1)\gamma^2\right\}\right] e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)(s^2-1)}}{\displaystyle\int_{-\frac{s-1}{2}}^{\frac{s-1}{2}} \left(\eta+\frac{s-1}{2}\right)\left(\eta-\frac{s-1}{2}\right)\left\{1-(s-1)\gamma-2\gamma\eta\right\}\left\{1+(s+1)\gamma-2\gamma\eta\right\} e^{-2\gamma\left(\frac{\theta}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}} d\eta}$$

$$= 2\, e^{\frac{\gamma}{2}\left(\frac{\theta}{T}\right)(s^2-1)} \int_0^{\left(\frac{s-1}{2}\right)} \left[4\gamma^2\eta^4 + \left\{1+2\gamma+4\gamma(1+\gamma)-2(s^2-1)\gamma^2\right\}\eta^2 - \frac{(s^2-1)}{4}\left\{1+2\gamma-(s^2-1)\gamma^2\right\}\right] e^{-2\gamma\left(\frac{\theta}{T}\right)\eta^2} d\eta$$

$$\int_0^{\frac{s-1}{2}} e^{-2\gamma\left(\frac{\theta}{T}\right)\eta^2} d\eta = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{2\gamma\frac{\theta}{T}}} \cdot \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{2\gamma\left(\frac{\theta}{T}\right)}\left(\frac{s-1}{2}\right)} e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{2\gamma\left(\frac{\theta}{T}\right)}} \operatorname{erf}\left(\sqrt{2\gamma\frac{\theta}{T}}\,\frac{s-1}{2}\right)$$

$$\int_0^{\frac{s-1}{2}} \eta^2 e^{-2\gamma\left(\frac{\theta}{T}\right)\eta^2} d\eta = -\frac{\partial}{\partial\left(2\gamma\frac{\theta}{T}\right)} \int_0^{\frac{s-1}{2}} e^{-2\gamma\left(\frac{\theta}{T}\right)\eta^2} d\eta$$

$$= \frac{\sqrt{\pi}}{4}\frac{1}{\left(2\gamma\frac{\theta}{T}\right)^{3/2}} \operatorname{erf}\left(\sqrt{2\gamma\frac{\theta}{T}}\,\frac{s-1}{2}\right) - \frac{s-1}{4}\frac{1}{\left(2\gamma\frac{\theta}{T}\right)} e^{-2\gamma\left(\frac{\theta}{T}\right)\left(\frac{s-1}{2}\right)^2}$$

$$\int_0^{\frac{s-1}{2}} \eta^4 e^{-2\gamma\left(\frac{\theta}{T}\right)\eta^2} d\eta = -\frac{\partial}{\partial\left(2\gamma\frac{\theta}{T}\right)} \int_0^{\frac{s-1}{2}} \eta^2 e^{-2\gamma\left(\frac{\theta}{T}\right)\eta^2} d\eta$$

$$= \frac{3\sqrt{\pi}}{8}\frac{1}{\left(2\gamma\frac{\theta}{T}\right)^{5/2}} \operatorname{erf}\left(\sqrt{2\gamma\frac{\theta}{T}}\,\frac{s-1}{2}\right) - \frac{3(s-1)}{8}\frac{1}{\left(2\gamma\frac{\theta}{T}\right)^2} e^{-2\gamma\left(\frac{\theta}{T}\right)\left(\frac{s-1}{2}\right)^2} - \frac{(s-1)^3}{16}\frac{1}{2\gamma\frac{\theta}{T}} e^{-2\gamma\left(\frac{\theta}{T}\right)\left(\frac{s-1}{2}\right)^2}$$

$$\int_{-\frac{s-1}{2}}^{\frac{s-1}{2}} \left(\eta+\frac{s-1}{2}\right)\left(\eta-\frac{s+1}{2}\right)\left\{1-(s-1)\gamma-2\gamma\eta\right\}\left\{1+(s+1)\gamma-2\gamma\eta\right\} e^{-2\gamma\left(\frac{h}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}} d\eta$$

$$= e^{-\frac{1}{2}\left(\frac{h}{T}\right)(s^2-1)} \Bigg[\gamma^2\left\{\frac{3\sqrt{\pi}}{\left(2\gamma\frac{h}{T}\right)^{5/2}}\,\mathrm{erf}\left(\sqrt{2\gamma\tfrac{h}{T}}\,\frac{s-1}{2}\right)-\left[\frac{3(s-1)}{\left(2\gamma\frac{h}{T}\right)^2}+\frac{(s-1)^3}{2}\frac{1}{\left(2\gamma\frac{h}{T}\right)}\right]e^{-2\gamma\left(\frac{h}{T}\right)\left(\frac{s-1}{2}\right)^2}\right\}$$

$$+\frac{1}{2}\left\{1+2\gamma+k\gamma(1+\gamma)-2(s^2-1)\gamma^2\right\}\left\{\frac{\sqrt{\pi}}{\left(2\gamma\frac{h}{T}\right)^{3/2}}\,\mathrm{erf}\left(\sqrt{2\gamma\tfrac{h}{T}}\,\frac{s-1}{2}\right)-\frac{(s-1)}{2\gamma\frac{h}{T}}e^{-2\gamma\left(\frac{h}{T}\right)\left(\frac{s-1}{2}\right)^2}\right\}$$

$$-\frac{(s^2-1)}{4}\left\{1+2\gamma-(s^2-1)\gamma^2\right\}\frac{\sqrt{\pi}}{\left(2\gamma\frac{h}{T}\right)^{1/2}}\,\mathrm{erf}\left(\sqrt{2\gamma\tfrac{h}{T}}\,\frac{s-1}{2}\right)\Bigg]$$

$$\int_{\frac{s-1}{2}}^{\frac{s+1}{2}} \left(\eta+\frac{s-1}{2}\right)\left(\eta-\frac{s+1}{2}\right)\left\{1-(s-1)\gamma-2\gamma\eta\right\}\left\{1+(s+1)\gamma-2\gamma\eta\right\} e^{-2\gamma\left(\frac{h}{T}\right)\left\{\eta^2+\frac{s^2-1}{4}\right\}} d\eta$$

$$= \int_0^1 (s-1+\xi)(\xi-1)\left\{1-2(s-1)\gamma-2\gamma\xi\right\}\left\{1+2\gamma-2\gamma\xi\right\} e^{-2\gamma\left(\frac{h}{T}\right)\left\{\frac{s(s-1)}{2}+(s-1)\xi+\xi^2\right\}} d\xi$$

$$\cong e^{-\gamma\left(\frac{h}{T}\right)s(s-1)}\int_0^1 (\xi-1)(\xi+s-1)\left\{1-2(s-1)\gamma\cancel{-2\gamma\xi}\right\}\left\{1\cancel{+2\gamma}\cancel{-2\gamma\xi}\right\} e^{-2\gamma\left(\frac{h}{T}\right)(s-1)\xi} d\xi$$

$$= e^{-\gamma\left(\frac{h}{T}\right)s(s-1)}\left\{1-2(s-1)\gamma\right\}\int_0^1\left\{\xi^2+(s-2)\xi-(s-1)\right\} e^{-2\gamma\left(\frac{h}{T}\right)(s-1)\xi} d\xi$$

$$\int_0^1 e^{-a\xi}d\xi = \frac{1}{a}\left[1-e^{-a}\right]$$

$$\int_0^1 \xi e^{-a\xi}d\xi = -\frac{\partial}{\partial a}\int_0^1 e^{-a\xi}d\xi = \frac{1}{a^2}(1-e^{-a})-\frac{1}{a}e^{-a}$$

$$\int_0^1 \xi^2 e^{-a\xi}d\xi = -\frac{\partial}{\partial a}\int_0^1 \xi e^{-a\xi}d\xi = +\frac{2}{a^3}(1-e^{-a})-\frac{2}{a^2}e^{-a}-\frac{1}{a}e^{-a}$$

$$\int_{\frac{s-1}{2}}^{\frac{s+1}{2}} \cdots \, d\eta = \left\{1-2(s-1)\gamma\right\} e^{-\gamma\left(\frac{b}{T}\right)s(s-1)} \left[\frac{2}{\left\{2\gamma\frac{b}{T}(s-1)\right\}^3} \left\{1 - e^{-2\gamma\left(\frac{b}{T}\right)(s-1)}\right\} \right.$$

$$- \frac{2}{\left\{2\gamma\frac{b}{T}(s-1)\right\}^2} e^{-2\gamma\left(\frac{b}{T}\right)(s-1)} - \frac{1}{2\gamma\left(\frac{b}{T}\right)(s-1)} e^{-2\gamma\left(\frac{b}{T}\right)(s-1)}$$

$$+ \frac{(s-2)}{\left\{2\gamma\frac{b}{T}(s-1)\right\}^2} \left\{1 - e^{-2\gamma\left(\frac{b}{T}\right)(s-1)}\right\} - \frac{s-2}{\left\{2\gamma\frac{b}{T}(s-1)\right\}} e^{-2\gamma\left(\frac{b}{T}\right)(s-1)} - \frac{1}{2\gamma\frac{b}{T}} \left\{1 - e^{-2\gamma\frac{b}{T}(s-1)}\right\}$$

$$S(s) \cong \frac{1}{4}\sqrt{\frac{\pi}{2\gamma}\left(\frac{T}{b}\right)}\, e^{-\frac{\gamma}{2}\left(\frac{b}{T}\right)(s^2-1)} \left[3\left(\frac{T}{b}\right)^2 \left\{1 - 2\,\mathrm{erf}\left(\sqrt{2\gamma\left(\frac{b}{T}\right)}\,\frac{s-1}{2}\right)\right\} \right.$$

$$\left. + \frac{1-2s^2\gamma^2}{\gamma}\left(\frac{T}{b}\right) \left\{ \left\{1 - 2\,\mathrm{erf}\left(\sqrt{2\gamma\left(\frac{b}{T}\right)}\,\frac{s-1}{2}\right)\right\} - (s^2-1)\left\{1 - s^2\gamma^2\right\}\left(1 - 2\,\mathrm{erf}\left(\sqrt{2\gamma\frac{b}{T}}\,\frac{s-1}{2}\right)\right) \right\} \right]$$

$$+ \frac{1}{2} e^{-\gamma\left(\frac{b}{T}\right)s(s-1)} \left[3(s-1)\left(\frac{T}{b}\right)^2 + \gamma(s-1)^3\left(\frac{T}{b}\right) + \frac{(1-2s^2\gamma^2)(s-1)}{\gamma}\left(\frac{T}{b}\right) \right]$$

$$- 2(1-2s\gamma)\, e^{-\gamma\left(\frac{b}{T}\right)s(s-1)} \left[\frac{2}{\left\{2\gamma\frac{b}{T}(s-1)\right\}^3} \left\{ 1 - \left[1 + 2\gamma\frac{b}{T}(s-1) + \frac{1}{2}\left(2\gamma\frac{b}{T}(s-1)\right)^2\right] e^{-2\gamma\left(\frac{b}{T}\right)(s-1)} \right\} \right.$$

$$\left. + \frac{(s-2)}{\left\{2\gamma\frac{b}{T}(s-1)\right\}^2} \left\{ 1 - \left[1 + 2\gamma\frac{b}{T}(s-1)\right] e^{-2\gamma\left(\frac{b}{T}\right)(s-1)} \right\} - \frac{1}{2\gamma\frac{b}{T}} \left\{1 - e^{-2\gamma\frac{b}{T}(s-1)}\right\} \right]$$

$$- \frac{1}{6} s \left\{ (1-2\gamma s)\, e^{-\gamma\left(\frac{b}{T}\right)s(s+1)} + (1+2\gamma s)\, e^{-\gamma\left(\frac{b}{T}\right)s(s-1)} \right\}$$

Now
$$\sum_{s=1}^{\infty} \frac{1}{1+\left(\frac{B_0}{\alpha}\right)^2 s^2} = \sum_{s=1}^{\infty} \frac{\left(\frac{\pi\alpha}{B_0}\right)^2}{\left(\frac{\pi\alpha}{B_0}\right)^2 + \pi^2 s^2}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi\alpha}{B_0}\right)\coth\left(\frac{\pi\alpha}{B_0}\right) - 1\right\} \cong \frac{1}{6}\left(\frac{\pi\alpha}{B_0}\right)^2 + \cdots$$

<u>Therefore</u>, for $B_0/\alpha \gg 1$,

$$\boxed{\int_{-\infty}^{\infty} |\rho L \cdot P_\nu|^2 \, d\left(\frac{\nu}{\nu^*}\right) \cong \frac{1}{8\sqrt{\pi}} \sqrt[4]{\frac{T}{2\gamma\theta}} \left[1 + \frac{1}{8}\frac{T}{\theta} + 3\left(\frac{T}{\theta}\right)^2\right] \left(\frac{\pi N_T \varepsilon^2 \rho L}{3\mu c^2 Q_{njm} \nu^*}\right)^2}$$

Approximating Function

$$\int_0^z (1 - e^{-t} - at + bt^2)\, t\, dt = 0$$

$$\int_0^z (1 - e^{-t} - at + bt^2)\, t^2\, dt = 0$$

But
$$\int_0^z t\, e^{-t}\, dt = \left[-e^{-t} t\right]_0^z + \int_0^z e^{-t}\, dt$$

$$= \left[-e^{-t} t - e^{-t}\right]_0^z = 1 - (1+z)e^{-z}$$

$$\int_0^z t^2 e^{-t}\, dt = \left[-e^{-t} t^2\right]_0^z + 2\int_0^z t\, e^{-t}\, dt$$

$$= -e^{-z} z^2 + 2 - 2(1+z)e^{-z} = 2 - 2\left(1 + z + \tfrac{1}{2}z^2\right)e^{-z}$$

So
$$\frac{1}{2}z^2 - 1 + (1+z)e^{-z} - \frac{a}{3}z^3 + \frac{b}{4}z^4 = 0$$

$$\frac{1}{3}z^3 - 2 + 2\left(1 + z + \frac{z^2}{2}\right)e^{-z} - \frac{a}{4}z^4 + \frac{b}{5}z^5 = 0$$

$$\boxed{\frac{z^3}{3}a - \frac{z^4}{4}b = \frac{1}{2}z^2 + (1+z)e^{-z} - 1}$$

$$\boxed{\frac{z^4}{4}a - \frac{z^5}{5}b = \frac{1}{3}z^3 + 2\left(1 + z + \frac{z^2}{2}\right)e^{-z} - 2}$$

$$a = \frac{\frac{z^5}{5}\left\{\frac{1}{2}z^2 + (1+z)e^{-z} - 1\right\} - \frac{z^4}{4}\left\{\frac{1}{3}z^3 + 2\left(1 + z + \frac{z^2}{2}\right)e^{-z} - 2\right\}}{z^8\left(\frac{1}{15} - \frac{1}{16}\right)}$$

$$= \frac{z^3\left(\frac{1}{10} - \frac{1}{12}\right) + \left\{\frac{z}{5}(1+z) - \frac{1}{2}\left(1 + z + \frac{z^2}{2}\right)\right\}e^{-z} + \frac{1}{2} - \frac{z}{5}}{z^4\left(\frac{1}{15} - \frac{1}{16}\right)}$$

$$b = \frac{-\frac{\cancel{z^3}}{3}\left\{\frac{1}{3}z^3 + 2\left(1+z+\frac{z^2}{2}\right)e^{-z} - 2\right\} + \frac{\cancel{z^4}}{4}\left\{\frac{1}{2}z^2 + (1+z)e^{-z} - 1\right\}}{z^5\left(\frac{1}{15} - \frac{1}{16}\right)}$$

$$= \frac{z^3\left(\frac{1}{8} - \frac{1}{9}\right) + \left\{\frac{z}{4}(1+z) - \frac{z}{3}\left(1+z+\frac{z^2}{2}\right)\right\}e^{-z} + \frac{z}{3} - \frac{z}{4}}{z^5\left(\frac{1}{15} - \frac{1}{16}\right)}$$

Or

$$\boxed{a = 4\frac{1}{z} - 12\left(\frac{10}{z^4} + \frac{6}{z^3} + \frac{1}{z^2}\right)e^{-z} + \frac{120}{z^4} - \frac{48}{z^3}}$$

$$\boxed{b = \frac{10}{3}\frac{1}{z^2} - 20\left(\frac{8}{z^5} + \frac{5}{z^4} + \frac{1}{z^3}\right)e^{-z} + \frac{160}{z^5} - \frac{60}{z^4}}$$

The highest value $P_\nu pL$ is given approximately by

$$z = \frac{\alpha N_T \varepsilon^2 pL}{3\mu c^2 Q_{njm} \nu^{*2}} \frac{\nu^{*2}}{\alpha^2} \left\{ j(1+2\gamma j)\, e^{-\gamma\left(\frac{b}{T}\right)j(j-1)} \right\}_{max.}$$

$$\boxed{z = \left(\frac{\pi N_T \varepsilon^2 pL}{3\mu c^2 Q_{njm} \nu^*}\right)\left(\frac{\nu^*}{\pi\alpha}\right) \left\{ j(1+2\gamma j)\, e^{-\gamma\left(\frac{b}{T}\right)j(j-1)} \right\}_{max}}$$

j is given by

$$1 + 4\gamma j - \gamma\left(\frac{b}{T}\right) j(1+2\gamma j)(2j-1) = 0$$

Or approximately

$$j = \frac{1}{\sqrt{2\gamma\left(\frac{b}{T}\right)}}$$

$$z = \frac{\pi N_T \varepsilon^2 pL}{3\mu c^2 Q_{njm} \nu^*}\left(\frac{\nu^*}{\pi\alpha}\right)\frac{1}{\sqrt{2\gamma\left(\frac{b}{T}\right)}}\, e^{-\frac{1}{2}}$$

$$\boxed{z = \frac{1}{\pi e^{\frac{1}{2}}}\, \frac{\pi N_T \varepsilon^2 pL}{3\mu c^2 Q_{njm} \nu^*}\left(\frac{\nu^*}{\alpha}\right)\sqrt{\frac{T}{2\gamma b}}}$$

$$\boxed{\begin{aligned}\varepsilon \cong \frac{15}{\pi^4}\frac{\left(\frac{\theta}{T}\right)^4}{\left(e^{\frac{\theta}{T}}-1\right)}\Bigg[&\left(\frac{1}{3}+\frac{T/\theta}{\gamma}\right)\frac{4\,\pi e^{\frac{1}{2}}}{\left(\frac{\nu^4}{\alpha}\right)\sqrt{\frac{T}{2\gamma\theta}}}\left\{1-3\left(\frac{10}{z^3}+\frac{6}{z^2}+\frac{1}{z}\right)e^{-z}+\frac{30}{z^3}-\frac{12}{z^2}\right\}\\ &-\frac{1}{8\sqrt{\pi}}\sqrt{\frac{T}{2\gamma\theta}}\left\{1+\frac{1}{\gamma}\frac{T}{\theta}+3\left(\frac{T}{\theta}\right)^2\right\}\frac{\pi^2 e}{\left(\frac{\nu^4}{\alpha}\right)\frac{T}{2\gamma\theta}}\left\{\frac{10}{3}-20\left(\frac{8}{z^3}+\frac{5}{z^2}+\frac{1}{z}\right)e^{-z}\right.\\ &\left.+\frac{160}{z^3}-\frac{60}{z^2}\right\}\Bigg].\end{aligned}}$$

$$\begin{aligned}\varepsilon \cong \frac{15}{\pi^3}\frac{\left(\frac{\theta}{T}\right)^4 e^{\frac{1}{2}}}{\left(e^{\theta/T}-1\right)}\frac{(\alpha/\nu^4)}{\sqrt{\frac{1}{2\gamma}\frac{T}{\theta}}}\Bigg[&\left(\frac{4}{3}+4\,\frac{T/\theta}{\gamma}\right)\left\{1-3\left(\frac{10}{z^3}+\frac{6}{z^2}+\frac{1}{z}\right)e^{-z}+\frac{30}{z^3}-\frac{12}{z^2}\right\}\\ &-\frac{e^{\frac{1}{2}}\sqrt{\pi}}{8}\left(1+\frac{1}{\gamma}\frac{T}{\theta}+3\frac{T^2}{\theta^2}\right)\left\{\frac{10}{3}-20\left(\frac{8}{z^3}+\frac{5}{z^2}+\frac{1}{z}\right)e^{-z}+\frac{160}{z^3}-\frac{60}{z^2}\right\}\Bigg]\end{aligned}$$

$$E(n,j)' = \frac{1}{2}h\nu_e - x_e\frac{1}{4}h\nu_e + E(n,j)$$

$$= \frac{1}{2}h\nu_e\left\{1-\frac{x_e}{2}\right\} + E(n,j)$$

$$\Delta E = E' - E = \frac{1}{2}h\nu_e\left(1-\frac{x_e}{2}\right)$$

$$\frac{\Delta E}{kT} = \frac{1}{2}\frac{h\nu_e}{kT}\left(1-\frac{x_e}{2}\right) = \frac{1}{2}\frac{\theta}{T}\left(1-\frac{x_e}{2}\right)$$

So

$$\boxed{z = \frac{N_T\,\varepsilon^2 pL}{3\mu c^2 Q_{njm}\,\alpha}\sqrt{\frac{T}{2\gamma\theta}}\; e^{-\frac{1}{2}\left\{\frac{\theta}{T}\left(1-\frac{x_e}{2}\right)+1\right\}}}$$

Corrections in F and F' 1

Emissivity of Diatomic Gases

I. Formulation

$$\boxed{\varepsilon = \dfrac{\displaystyle\int_0^\infty \frac{\nu^3 \left\{1 - e^{-P_\nu p L}\right\} d\nu}{e^{\frac{k}{T}\frac{\nu}{\nu^*}} - 1}}{\displaystyle\int_0^\infty \frac{\nu^3\, d\nu}{e^{\frac{k}{T}\frac{\nu}{\nu^*}} - 1}}}$$

$$P_\nu \cong \frac{\alpha}{\pi} \sum_{j=1}^{\infty} \left[\frac{S_{j\to j-1}^{0\to 1}}{\left(\nu - \nu_{j\to j-1}^{0\to 1}\right)^2 + \alpha^2} + \frac{S_{j-1\to j}^{0\to 1}}{\left(\nu - \nu_{j-1\to j}^{0\to 1}\right)^2 + \alpha^2} \right]$$

where

$$S_{j\to j-1}^{0\to 1} = \frac{N_T \varepsilon^2 \pi}{3\mu c^2 Q_{njm}} \, \frac{\nu_{j\to j-1}^{0\to 1}}{\nu_{0\to 0}^{0\to 1}} \, j\, e^{-\frac{E_{0,j}}{kT}} F \cdot G$$

$$S_{j-1\to j}^{0\to 1} = \frac{N_T \varepsilon^2 \pi}{3\mu c^2 Q_{njm}} \, \frac{\nu_{j-1\to j}^{0\to 1}}{\nu_{0\to 0}^{0\to 1}} \, j\, e^{-\frac{E_{0,j-1}}{kT}} F' \cdot G'$$

$$F = 1 + 8\gamma j \left(1 + \frac{5\gamma j}{4} - \frac{3\gamma}{4}\right)$$

$$F' = 1 - 8\gamma j \left(1 - \frac{5\gamma j}{4} - \frac{3\gamma}{4}\right)$$

$$G = 1 - \exp\left\{-\left(\frac{hc}{kT}\right) \nu_{j\to j-1}^{0\to 1}\right\}$$

$$G' = 1 - \exp\left\{-\left(\frac{hc}{kT}\right) \nu_{j-1\to j}^{0\to 1}\right\}$$

$$\nu^{0\to 1}_{j\to j-1} = \frac{1}{hc}\left| E_{0,j} - E_{1,j-1}\right|$$

$$\nu^{0\to 1}_{j-1\to j} = \frac{1}{hc}\left| E_{0,j-1} - E_{1,j}\right|$$

$$\nu^{0\to 1}_{0\to 0} = \frac{1}{hc}\left| E_{0,0} - E_{1,0}\right|$$

$$E_{n,j} = (k\theta)\left[n - xn(n-1) + \gamma j(j+1)\left\{1 - 4\gamma^2 j(j+1) - \delta n\right\}\right]$$

$$\gamma = \frac{B_0}{\nu^*}, \quad \delta \cong 6\gamma\left[\left(\frac{x}{\gamma}\right)^{1/2} - 1\right],$$

$$Q_{njm} = \frac{\frac{T}{\gamma\theta}}{(1-e^{-\theta/T})}\left[1 + \gamma\left\{\frac{1}{3}\left(\frac{\theta}{T}\right) + \delta\left(\frac{T}{\theta}\right)\right\} + \frac{\delta}{e^{\theta/T}-1} + \frac{2x\,\frac{\theta}{T}}{(e^{\theta/T}-1)^2}\right]$$

Let $\dfrac{N_1\varepsilon^2\pi}{3\mu c^2} = \beta\left(\frac{\theta}{T}\right)$

$$\beta = \frac{N_1\varepsilon^2\pi}{3\mu c^2}$$

$$\nu^{0\to 1}_{j\to j-1} = \frac{k\theta}{hc}\left[1 + \gamma j(j-1)\left\{1 - 4\gamma^2 j(j-1) - \delta\right\} - \gamma j(j+1)\left\{1 - 4\gamma^2 j(j+1)\right\}\right]$$

$$= \frac{k\theta}{hc}\left[1 - \gamma j\left\{2 - 16\gamma^2 j^2 + \delta(j-1)\right\}\right]$$

$$\nu^{0\to 1}_{j-1\to j} = \frac{k\theta}{hc}\left[1 + \gamma j(j+1)\left\{1 - 4\gamma^2 j(j+1) - \delta\right\} - \gamma j(j-1)\left\{1 - 4\gamma^2 j(j-1)\right\}\right]$$

$$= \frac{k\theta}{hc}\left[1 + \gamma j\left\{2 - 16\gamma^2 j^2 - \delta(j+1)\right\}\right]$$

Hence
$$\frac{\nu^{0\to 1}_{j\to j-1}}{\nu^{0\to 1}_{0\to 0}} = 1 - 2\gamma j\left\{1 - 8\gamma^2 j^2 + \frac{\xi}{2}(j-1)\right\}$$

$$\frac{\nu^{0\to 1}_{j-1\to j}}{\nu^{0\to 1}_{0\to 0}} = 1 + 2\gamma j\left\{1 - 8\gamma^2 j^2 - \frac{\xi}{2}(j+1)\right\}$$

$$\frac{hc}{kT}\nu^{0\to 1}_{j\to j-1} = \frac{\theta}{T}\left[1 - 2\gamma j\left\{1 - 8\gamma^2 j^2 + \frac{\xi}{2}(j-1)\right\}\right]$$

$$\frac{hc}{kT}\nu^{0\to 1}_{j-1\to j} = \frac{\theta}{T}\left[1 + 2\gamma j\left\{1 - 8\gamma^2 j^2 - \frac{\xi}{2}(j+1)\right\}\right]$$

$$G = 1 - e^{-\frac{\theta}{T}}\left[1 + 2\gamma\left(\frac{\theta}{T}\right)j\left\{1 - 8\gamma^2 j^2 + \frac{\xi}{2}(j-1)\right\} + 2\gamma^2\left(\frac{\theta}{T}\right)^2 j^2 \cdots\right]$$

$$\boxed{G = \left(1 - e^{-\frac{\theta}{T}}\right)\left[1 + \frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1 - e^{-\theta/T}}\gamma j\left\{1 + \frac{\xi}{2}(j-1) + \gamma\left(\frac{\theta}{T}\right)j\right\}\cdots\right]}$$

$$\boxed{G' = \left(1 - e^{-\frac{\theta}{T}}\right)\left[1 - \frac{2\left(\frac{\theta}{T}\right)e^{-\theta/T}}{1 - e^{-\theta/T}}\gamma j\left\{1 - \frac{\xi}{2}(j+1) - \gamma\left(\frac{\theta}{T}\right)j\right\}\cdots\right]}$$

$$FG = \left(1 - e^{-\frac{\theta}{T}}\right)\left[1 + \frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1 - e^{-\theta/T}}\gamma j\left\{1 + \frac{\xi}{2}(j-1) + \gamma\left(\frac{\theta}{T}\right)j\right\}\cdots\right]\left[1 + 8\gamma j\left(1 + \frac{5\gamma j}{4} - \frac{3\gamma}{4}\right)\right]$$

$$F'G' = \left(1 - e^{-\frac{\theta}{T}}\right)\left[1 - \frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1 - e^{-\theta/T}}\gamma j\left\{1 - \frac{\xi}{2}(j+1) - \gamma\left(\frac{\theta}{T}\right)j\right\}\cdots\right]\left[1 - 8\gamma j\left(1 - \frac{5\gamma j}{4} - \frac{3\gamma}{4}\right)\right]$$

II. Approximation for non-overlapping lines

$$\varepsilon = \frac{60}{\pi^4}\frac{\left(\frac{\theta}{T}\right)^4}{\left(e^{\frac{\theta}{T}}-1\right)}\sqrt{\frac{\alpha\beta\left(\frac{\theta}{T}\right)pL}{\pi\, Q_{vjm}\,\nu^{*2}C}}\;\sum_{j=1}^{\infty}\left\{\sqrt{\frac{\nu^{0\to1}_{j\to j-1}}{\nu^{0\to1}_{0\to0}}\, j\, e^{-\frac{E_{0,j}}{kT}}F G} + \sqrt{\frac{\nu^{0\to1}_{j-1\to j}}{\nu^{0\to1}_{0\to0}}\, j\, e^{-\frac{E_{0,j-1}}{kT}}F' G'}\right\}$$

therefore

$$\varepsilon = \frac{60}{\pi^4}\frac{\left(\frac{\theta}{T}\right)^4}{\left(e^{\frac{\theta}{T}}-1\right)}\left\{\frac{\left(\frac{\theta}{T}\right)\left(1-e^{-\theta/T}\right)^2}{\frac{T}{\gamma\theta}\left[1+\gamma\left(\frac{1}{3}\frac{\theta}{T}+\delta\frac{T}{\theta}\right)+\frac{\epsilon}{e^{\theta/T}-1}+\frac{2x\frac{\theta}{T}}{\left(e^{\theta/T}-1\right)^2}\right]}\;\frac{\alpha\beta pL}{\pi\nu^{*2}C}\right\}^{1/2}$$

$$\cdot\sum_{j=1}^{\infty}\Bigg[\Big\{j\left[1-2\gamma j\left\{1-8\gamma^2 j^2+\frac{\epsilon}{2}(j-1)\right\}\right]\left[1+\frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\gamma j\left\{1-\frac{\epsilon}{2}(j-1)+\gamma\left(\frac{\theta}{T}\right)j\right\}\cdot\right]\left[1+8\gamma j\left(1+\frac{5\gamma j}{4}-\frac{3\epsilon}{4}\right)\right]\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j(j+1)\left\{1-4\gamma^2 j(j+1)\right\}}\Big\}^{1/2}$$

$$+\Big\{j\left[1+2\gamma j\left\{1-8\gamma^2 j^2-\frac{\epsilon}{2}(j+1)\right\}\right]\left[1-\frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\gamma j\left\{1-\frac{\epsilon}{2}(j+1)-\gamma\left(\frac{\theta}{T}\right)j\right\}\cdot\right]\left[1-8\gamma j\left(1-\frac{5\gamma j}{4}-\frac{3\epsilon}{4}\right)\right]\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j(j-1)\left\{1-4\gamma^2 j(j-1)\right\}}\Big\}^{1/2}$$

$$\boxed{\begin{aligned}\varepsilon &= \frac{60}{\pi^4} e^{-\frac{\theta}{T}} \left(\frac{\theta}{T}\right)^5 \sqrt{\frac{\gamma\alpha}{C v^2}} \sqrt{\frac{\beta p L}{\pi v^*}} \left[1 - \gamma\left(\frac{1}{6}\frac{\theta}{T} + 4\frac{T}{\theta}\right) - \frac{\delta/2}{e^{\theta/T}-1} - \frac{x\frac{\theta}{T}}{(e^{\theta/T}-1)^2}\right] \\ &\qquad S\left(\gamma, \delta, \frac{\theta}{T}\right)\end{aligned}}$$

$$\left[1 - 2\gamma j\left\{1 - \delta\gamma^2 j^2 + \frac{\delta}{2}(j-1)\right\}\right]\left[1 + \frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\gamma j\left\{1 + \frac{\delta}{2}(j-1) + \gamma\left(\frac{\theta}{T}\right)j\right\}\cdots\right]$$

$$\left[1 + 8\gamma j\left(1 + \frac{5\gamma j}{4} - \frac{3\gamma}{4}\right)\right]$$

$$= \left[1 - 2\left(1 - \frac{\delta}{2}\right)\gamma j - \delta\gamma j^2 \cdots\right]\left[1 + \frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\left(1-\frac{\delta}{2}\right)\gamma j + \frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\left(\frac{\delta}{2} + \gamma\frac{\theta}{T}\right)\gamma j^2 \cdots\right]$$

$$\left[1 + 8\left(1 - \frac{3\gamma}{4}\right)\gamma j + 10\gamma^2 j^2 \cdots\right]$$

$$= 1 + \left\{8\left(1 - \frac{3\gamma}{4}\right) + \frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\left(1 - \frac{\delta}{2}\right) - 2\left(1 - \frac{\delta}{2}\right)\right\}\gamma j$$

$$+ \left\{10 + \frac{2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\left(\frac{\delta}{2\gamma} + \frac{\theta}{T}\right) - \frac{\delta}{\gamma} - \frac{4\left(1-\frac{\delta}{2}\right)^2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}} - 16\left(1-\frac{\delta}{2}\right)\left(1 - \frac{3\gamma}{4}\right)\right.$$

$$\left. + \frac{16\left(1-\frac{\delta}{2}\right)\left(1-\frac{3\gamma}{4}\right)\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\right\}\gamma^2 j^2 \cdots$$

$$\left\{[\quad][\quad][\quad]\right\}^{1/2} = 1 + \left\{4\left(1 - \frac{3\gamma}{4}\right) + \frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}\left(1-\frac{\delta}{2}\right)}{1-e^{-\theta/T}} - \left(1 - \frac{\delta}{2}\right)\right\}\gamma j$$

$$+ \left\{5 + \frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}\left(\frac{\delta}{2\gamma} + \frac{\theta}{T}\right)}{1-e^{-\theta/T}} - \frac{\delta}{2\gamma} - \frac{2\left(1-\frac{\delta}{2}\right)^2\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}} - 8\left(1-\frac{\delta}{2}\right)\left(1-\frac{3\gamma}{4}\right) + \frac{8\left(1-\frac{\delta}{2}\right)\left(1-\frac{3\gamma}{4}\right)\frac{\theta}{T}e^{-\theta/T}}{1-e^{-\theta/T}}\right.$$

$$\left. - \frac{1}{2}\left[4\left(1-\frac{3\gamma}{4}\right) + \frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}\left(1-\frac{\delta}{2}\right)}{1-e^{-\theta/T}} - \left(1-\frac{\delta}{2}\right)\right]^2\right\}\gamma^2 j^2 + \cdots$$

$$e^{-\frac{\gamma}{2}\frac{\theta}{T} j(j+1)}\left\{1-4\gamma^2 j(j+1)\right\} = e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)(j^2+j)}\left\{1-4\gamma^2 j(j+1)\right\}$$

$$= e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2 - \frac{\gamma}{2}\left(\frac{\theta}{T}\right)j} \cdots = e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2}\left\{1-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j+\frac{1}{8}\left(\frac{\theta}{T}\right)^2\gamma^2 j^2 \cdots\right\}$$

$\therefore$ the significant terms are

$$1+\Bigg\{\frac{5}{4}+\frac{\left(\frac{\theta}{T}\right)e^{-\theta/T}}{1-e^{-\theta/T}}\left[\frac{\delta}{2\gamma}+\frac{\theta}{T}-2\left(1-\frac{\delta}{2}\right)^2+8\left(1-\frac{\delta}{2}\right)\left(1-\frac{3\gamma}{4}\right)\right]-\frac{\delta}{2\gamma}-8\left(1-\frac{\delta}{2}\right)\left(1-\frac{3\gamma}{4}\right)$$

$$-\frac{1}{2}\left[4\left(1-\frac{3\gamma}{4}\right)-\left(1-\frac{\delta}{2}\right)+\frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}\left(1-\frac{\delta}{2}\right)}{1-e^{-\theta/T}}\right]^2+\frac{1}{8}\left(\frac{\theta}{T}\right)^2$$

$$-\frac{1}{2}\left(\frac{\theta}{T}\right)\left[4\left(1-\frac{3\gamma}{4}\right)-\left(1-\frac{\delta}{2}\right)+\frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}\left(1-\frac{\delta}{2}\right)}{1-e^{-\theta/T}}\right]\Bigg\}\gamma^2 j^2$$

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$$=1+\Bigg\{5+\frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\left[\frac{\delta}{2\gamma}+\frac{\theta}{T}+6\right]-\frac{\delta}{2\gamma}-8-\frac{1}{2}\left[3+\frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\right]^2+\frac{1}{8}\left(\frac{\theta}{T}\right)^2$$

$$-\frac{1}{2}\left(\frac{\theta}{T}\right)\left[3+\cancel{\frac{\left(\frac{\theta}{T}\right)e^{-\theta/T}}{1-e^{-\theta/T}}}\right]\Bigg\}\gamma^2 j^2$$

$$=1+\Bigg\{5+\frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\left[\frac{\delta}{2\gamma}+\frac{\theta}{T}-\frac{1}{2}\frac{\theta}{T}+6-3\right]-\frac{\delta}{2\gamma}-8-\frac{9}{2}-\frac{1}{2}\frac{\left(\frac{\theta}{T}\right)^2 e^{-2\frac{\theta}{T}}}{\left(1-e^{-\theta/T}\right)^2}+\frac{1}{8}\left(\frac{\theta}{T}\right)^2-\frac{3}{2}\frac{\theta}{T}\Bigg\}\gamma^2 j^2$$

$$=1+\Bigg\{\frac{1}{8}\left(\frac{\theta}{T}\right)^2-\frac{3}{2}\left(\frac{\theta}{T}\right)+\frac{\left(\frac{\theta}{T}\right)e^{-\frac{\theta}{T}}}{1-e^{-\theta/T}}\left[\frac{\delta}{2\gamma}+\frac{1}{2}\left(\frac{\theta}{T}\right)+3\right]-\frac{\delta}{2\gamma}-\frac{15}{2}-\frac{1}{2}\frac{\left(\frac{\theta}{T}\right)^2 e^{-2\frac{\theta}{T}}}{\left(1-e^{-\theta/T}\right)^2}\Bigg\}\gamma^2 j^2$$

$$S\left(\gamma,\delta,\frac{1}{T}\right) \cong -\frac{5}{12} + 2\int_0^\infty \sqrt{j}\, e^{-\frac{\gamma}{2}\left(\frac{1}{T}\right)j^2}\, dj$$

$$+2\left\{ \frac{1}{8}\left(\frac{1}{T}\right)^2 - \frac{3}{2}\left(\frac{1}{T}\right) + \frac{\left(\frac{1}{T}\right)e^{-1/T}}{1-e^{-1/T}}\left[\frac{5}{2\gamma}+\frac{1}{2}\left(\frac{1}{T}\right)+3\right] - \frac{5}{2\gamma} - \frac{18}{8} - \frac{1}{2}\,\frac{\left(\frac{1}{T}\right)^2 e^{-2\frac{1}{T}}}{\left(1-e^{-1/T}\right)^2}\right\}\int_0^\infty \gamma^2 j^{5/2} e^{-\frac{\gamma}{2}\left(\frac{1}{T}\right)j^2}\, dj$$

$$\cong -\frac{5}{12} + \left(\frac{2T}{\gamma\delta}\right)^{3/4}\int_0^\infty \eta^{(3/4)-1} e^{-\eta}\, d\eta + \left(\frac{2T}{\gamma\delta}\right)^{3/4}\gamma^2\,\frac{2T}{\gamma\delta}\left\{\quad\right\}\int_0^\infty \eta^{7/4-1} e^{-\eta}\, d\eta$$

$$\cong -\frac{5}{12} + \left(\frac{2T}{\gamma\delta}\right)^{3/4}\left[\Gamma\left(\frac{3}{4}\right) + \frac{2T}{\delta}\gamma\left\{\quad\right\}\Gamma\left(\frac{7}{4}\right)\right]$$

$$= -\frac{5}{12} + \Gamma\left(\frac{3}{4}\right)\left(\frac{2T}{\gamma\delta}\right)^{3/4}\left[1 + \frac{3}{2}\,\frac{\gamma T}{\delta}\left\{\quad\right\}\right]$$

ROUGH DRAFT 30 July 1951

EMISSIVITY OF DIATOMIC GASES AT LOW PRESSURES

I. INTRODUCTION

Emissivity calculations for diatomic gases from spectroscopic data were developed recently by S. S. Penner (Ref. 1). His method is based upon the use of an average absorption coefficient for the entire fundamental and higher vibration-rotation bands. The method is thus effective when there are extensive overlapping and broadening of the spectral lines, and hence is accurate for gases at high total pressures and temperatures. At low pressures, the lines do not overlap and a different approach to the problem should be made. Penner and M. H. Ostrander (Ref. 2) have computed the emissivity of carbon monoxide for the case of non-overlapping lines by a numerical procedure, using spectroscopic data obtained recently by Penner and D. Weber (Ref. 3). The results are in excellent agreement with the emissivity determined experimentally by W. Ullrich and H. C. Hottel (Ref. 4). The amount of numerical work involved is, however, rather heavy. It is the purpose of the present paper to develope an approximate but convenient formula for calculating the emissivity of diatomic gases for the case of non-overlapping lines.

II. FORMULATION OF THE PROBLEM

If T is the temperature, θ the characteristic temperature, ν the wave number, ν^* the characteristic wave number, P_ν the spectral absorption coefficient at ν, p the partial pressure of the emitting gas,

ω · wave numb

$= h\nu^*/k$

and L the optical path length, then the emissivity ε of the gas under the specified conditions is

$$\varepsilon = \int_0^\infty \frac{\nu^3\left\{1 - e^{-P_\nu pL}\right\} d\nu}{e^{\left(\frac{h}{k}\right)\frac{\nu}{T}} - 1} \Bigg/ \int_0^\infty \frac{\nu^3\, d\nu}{e^{\left(\frac{h}{k}\right)\frac{\nu}{T}} - 1} \tag{1}$$

at low temps where the contributions from upper harmonics is negligible

If only the fundamental vibration-rotational band is considered, the absorption coefficient P_ν is given approximately by

$$P_\nu = \frac{b}{\pi} \sum_{j=1}^{\infty} \left[\frac{S_{j\to j-1}^{0\to 1}}{(\nu - \nu_{j\to j-1}^{0\to 1})^2 + b^2} + \frac{S_{j-1\to j}^{0\to 1}}{(\nu - \nu_{j-1\to j}^{0\to 1})^2 + b^2} \right] \tag{2}$$

where b the half-width of the spectral lines, and Ss' are the integrated absorptions for the lines centering on the wave numbers corresponding to the indicated transitions. The Ss' can be computed in turn by using the results of J. R. Oppenheimer (Ref. 5). As

$$S_{j\to j-1}^{0\to 1} = \frac{N_T \epsilon^2 \pi}{3\mu c^2 Q} \frac{\nu_{j\to j-1}^{0\to 1}}{\nu^*} j e^{-\frac{E_{0,j}}{kT}} F\cdot G \tag{3}$$

This

and

$$S_{j-1\to j}^{0\to 1} = \frac{N_T \epsilon^2 \pi}{3\mu c^2 Q} \left(\frac{\nu_{j-1\to j}^{0\to 1}}{\nu^*}\right) j e^{-\frac{E_{0,j-1}}{kT}} F'\cdot G' \tag{4}$$

partition functions referred to zero as reference state. Thus should be multiplied by $\sim e^{-\varepsilon/2T}$

where N_T is the number of emitting molecules at temperature T per unit volume per unit pressure, ϵ the effective charge, μ the reduced mass, c the velocity of light, Q the complete internal partition function, E's are the internal energy levels given by

$$E_{v,j} = k\theta \left[v - xv(v-1) + \gamma j(j+1)\left\{1 - 4\gamma^2 j(j+1) - \delta v\right\}\right] \tag{5}$$

$E(v,j) - E(0,0)$

$x = x^*$

$\gamma \simeq Be/\omega e = \frac{1}{2}(De/Be)^{1/2}$; Note approximations

$\delta \simeq De/Be$

$\theta = hc\omega^*/k = h\nu^*/k$

~~(5)~~

where x, γ , δ are molecular constants in their standard notations. These constants are non-dimensional and are small. The F's and G's are

$$F(j,\gamma) = 1 + 4\gamma j\left(1 + \frac{5\gamma j}{8} - \frac{3\gamma}{8}\right)$$

$$F'(j,\gamma) = F(-j,\gamma) = 1 - 4\gamma j\left(1 - \frac{5\gamma j}{8} - \frac{3\gamma}{8}\right) \tag{6}$$

and

$$G = 1 - \exp\left\{-\left(\frac{hc}{kT}\right)\nu^{0\to 1}_{j\to j-1}\right\}$$

$$G' = 1 - \exp\left\{-\left(\frac{hc}{kT}\right)\nu^{0\to 1}_{j-1\to j}\right\} \tag{7}$$

The complete internal partition function can be written as

$$Q = \frac{1}{\gamma\frac{\theta}{T}\left(1-e^{-\theta/T}\right)}\left[1 + \gamma\left(\frac{1}{3}\frac{\theta}{T} + \delta\frac{T}{\theta}\right) + \frac{\delta}{e^{\theta/T}-1} + \frac{2x\frac{\theta}{T}}{\left(e^{\theta/T}-1\right)^2}\right] \tag{8}$$

If the fundamental vibration-rotation band gives the main contribution to the emissivity of the gas, the above equations give the necessary information to calculate approximately the emissivity ε .

III. APPROXIMATE SOLUTION

The numerical work is carrying out the computation indicated in the

previous section is very heavy. A short formula, however, can be developed: First of all, when the lines are ~~spearated~~ separated from each other, each line can be considered alone, independent of others. Furthermore, the value of the factor outside of the bracket in the numerator of Eq. (~~11~~ 7) can be approximated by its value at the center of each line. Thus according to S. S. Penner (Ref. 6)

9A

$$\varepsilon = \frac{15}{\pi^4}\left(\frac{\theta}{T}\right)^4 \sum_{j=1}^{\infty}\left[\frac{\left(\nu^{0\to1}_{j\to j-1}/\nu^*\right)^3}{e^{(\theta/T)(\nu^{0\to1}_{j\to j-1}/\nu^*)}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j\to j-1}\rho L}\right)d(\nu/\nu^*)\right.$$

$$\left.+\frac{\left(\nu^{0\to1}_{j-1\to j}/\nu^*\right)^3}{e^{(\theta/T)(\nu^{0\to1}_{j-1\to j}/\nu^*)}-1}\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j-1\to j}\rho L}\right)d(\nu/\nu^*)\right] \tag{9}$$

where the Ps are the absorption coefficient due to the particular line with transitions as indicated. The integrals can be easily evaluated (Ref. 6) and are given by the modified Bessel functions I_o and I_1:

$$\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j\to j-1}\rho L}\right)d(\nu/\nu^*) = 2\pi(b/\nu^*)\,\xi_j\,e^{-\xi_j}\left[I_0(\xi_j)+I_1(\xi_j)\right] \tag{10}$$

and

$$\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to1}_{j-1\to j}\rho L}\right)d(\nu/\nu^*) = 2\pi(b/\nu^*)\,\eta_j\,e^{-\eta_j}\left[I_0(\eta_j)+I_1(\eta_j)\right] \tag{11}$$

where

$$\xi_j = S^{0\to 1}_{j\to j-1}\, pL/2\pi b \tag{12}$$

and

$$\eta_j = S^{0\to 1}_{j-1\to j}\, pL/2\pi b \tag{13}$$

A further approximation can now be made. The magnitude of ξ's and η's are generally quite large if the product pL of pressure and optical path length is of the order of unity. Therefore, the asymptotic values of the Bessel functions can be used. Then

$$\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to 1}_{j\to j-1}pL}\right)d\left(\frac{\nu}{\nu^*}\right) \cong 2\sqrt{\frac{b\, S^{0\to 1}_{j\to j-1}\, pL}{\nu^{*2}}} \tag{14}$$

and

$$\int_{-\infty}^{\infty}\left(1-e^{-P^{0\to 1}_{j-1\to j}pL}\right)d\left(\frac{\nu}{\nu^*}\right) \cong 2\sqrt{\frac{b\, S^{0\to 1}_{j-1\to j}\, pL}{\nu^{*2}}} \tag{15}$$

By substituting Eqs. (14) and (15) into (9), the emissivity is calculated as a sum over j.

To carry out the sum over j, one can use the Euler-Maclaurin summation formula (Ref. 7), which evaluates the sum by an integral. First, due to the smallness of γ , δ , the following expansions, including terms up to the square of γj , are approximate.

$$\frac{\nu^{0\to 1}_{j\to j-1}}{\nu^*} = 1-2\gamma j-\left(\frac{\delta}{\gamma}\right)\gamma^2 j^2 \tag{16}$$

$$\sqrt{F} = 1 + \cancel{2}^{4}\gamma j - \cancel{2}^{3}\gamma j^2 \tag{17}$$

$$\frac{\sqrt{G}}{e^{\left(\frac{b}{T}\right)\left(\nu^{0\to 1}_{j\to j-1}/\nu^*\right)} - 1} = e^{-\left(\frac{b}{T}\right)\left(\nu^{0\to 1}_{j\to j-1}/\nu^*\right)}\left\{1 - e^{-\left(\frac{b}{T}\right)\left(\nu^{0\to 1}_{j\to j-1}/\nu^*\right)}\right\}^{-\frac{1}{2}}$$

The use of this quantity in the numerator is correct if the original form of Eq.(18) is used (i.e., ref. state in ground level).

$$= \frac{e^{-\frac{b}{T}}}{\left(1 - e^{-b/T}\right)^{1/2}}\left[1 + \left(\frac{b}{T}\right)\left\{2 + \frac{e^{-\frac{b}{T}}}{\left(1 - e^{-b/T}\right)}\right\}\gamma j + \left\{\left(\frac{b}{T}\right)\left(\frac{\delta}{\gamma} + 2\frac{b}{T}\right) + \frac{\left(\frac{b}{T}\right)e^{-\frac{b}{T}}}{\left(1 - e^{-b/T}\right)}\left(3\frac{b}{T} + \frac{\delta}{2\gamma}\right) + \frac{3}{2}\frac{\left(\frac{b}{T}\right)^2 e^{-2\frac{b}{T}}}{\left(1 - e^{-b/T}\right)^2}\right\}\gamma^2 j^2\right] \tag{18}$$

and

$$e^{-\frac{[E(0,j) - E(0,0)]}{2kT}} \qquad \cancel{e^{-\frac{E_{0j}}{2kT}}} = e^{-\frac{\gamma}{2}\frac{b}{T}j^2}\left[1 - \frac{1}{2}\left(\frac{b}{T}\right)\gamma j + \frac{1}{8}\left(\frac{b}{T}\right)^2\gamma^2 j^2\right] \tag{19}$$

The corresponding quantities for the transitions $j - 1 \to j$ can be very easily obtained from Eqs. (16) to (19) by replacing j with $-j$. Because of this property of symmetry, the sum of terms from the transition $j \to j - 1$ and the transition $j - 1 \to j$ for every j is a function of j^2 only. Thus, after appropriate canceling of linear terms,

$$\frac{\left(\nu^{0\to 1}_{j\to j-1}/\nu^*\right)^3}{e^{\left(\frac{b}{T}\right)\left(\nu^{0\to 1}_{j\to j-1}/\nu^*\right)} - 1}\int_{-\infty}^{\infty}\left(1 - e^{-P^{0\to 1}_{j\to j-1}kL}\right)d\left(\frac{\nu}{\nu^*}\right) + \frac{\left(\nu^{0\to 1}_{j-1\to j}/\nu^*\right)^3}{e^{\left(\frac{b}{T}\right)\left(\nu^{0\to 1}_{j-1\to j}/\nu^*\right)} - 1}\int_{-\infty}^{\infty}\left(1 - e^{-P^{0\to 1}_{j-1\to j}kL}\right)d\left(\frac{\nu}{\nu^*}\right)$$

$$= 4\left(\frac{b}{T}\right)e^{-\left(\frac{b}{T}\right)}f\left(\gamma, \delta, \frac{b}{T}\right)\sqrt{\left(\frac{\gamma b}{\nu^*}\right)\left(\frac{A\rho L}{\nu^*}\right)}\sqrt{j}\, e^{-\frac{\gamma}{2}\left(\frac{b}{T}\right)j^2}\left[1 + g\left(\gamma, \delta, \frac{b}{T}\right)\gamma^2 j^2\right] \tag{20}$$

where

$$A = \frac{N_I \epsilon^2 \pi}{3\mu c^2}\frac{I}{\theta} \qquad (21)$$

A is thus a constant independent of temperature and pressure. The f function is simply deduced from the ~~position~~ partition function Q as given by Eq. (8):

$$f = 1 - \gamma\left(\frac{1}{6}\frac{\theta}{T} + 4\frac{T}{\theta}\right) - \frac{\delta/\gamma}{\left(e^{\theta/T}-1\right)} - \frac{\gamma\frac{\theta}{T}}{\left(e^{\theta/T}-1\right)^2} \qquad (22)$$

Therefore f is a quantity close to unity. The function g is computed from the expansions given in Eqs. (16) to (~~20~~ 19). It is

$$g = \frac{3}{2}\left\{\frac{\frac{\theta}{T}e^{-\theta/T}}{1-e^{-\theta/T}}\right\}^2 + \frac{\frac{\theta}{T}e^{-\theta/T}}{1-e^{-\theta/T}}\left\{\frac{5}{2}\frac{\theta}{T} + \frac{5}{2\gamma}\right\} - 3 + \frac{\theta}{T}\left\{\frac{\delta}{\gamma} - \frac{9}{2} + \frac{7}{8}\frac{\theta}{T}\right\} - 15 - \frac{7}{3\gamma} \qquad (23)$$

The Euler-Mac~~ke~~laurin formula can be now employed to evaluate the sum in the emissivity ε . The resulting integral over j extends from 1 to ∞ . But this range can be made to be from 0 to ∞ by simply deducting the approximate value of the integral from 0 to 1 from the extended integral. Thus

$$2\sum_{j=1}^{\infty}\sqrt{j}\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2}\left[1 + g\gamma^2 j^2\right] \cong 2\int_1^{\infty}\sqrt{j}\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2}\left[1 + g\gamma^2 j^2\right]dj + \frac{11}{12}$$

$$\doteq 2\int_0^{\infty}\sqrt{j}\, e^{-\frac{\gamma}{2}\left(\frac{\theta}{T}\right)j^2}\left[1 + g\gamma^2 j^2\right]dj - \frac{5}{12}$$

$$= \Gamma\left(\tfrac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/4}\left[1 + \frac{3}{2}\frac{\gamma T}{\theta}g\right] - \frac{5}{12}$$

The $\Gamma\left(\frac{3}{4}\right)$ has the numerical value of 1.225. Finally then the expression

~~for~~ The emissivity for the case of non-overlapping lines is

$$\epsilon = \frac{30}{\pi^4}\left(\frac{\theta}{T}\right)^5 e^{-\theta/T} f\left(\gamma, \delta, x, \frac{\theta}{T}\right)\left[\Gamma\left(\frac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/4}\left\{1 + \frac{1}{2}\frac{\gamma T}{\theta} g\left(\gamma, \delta, x, \frac{\theta}{T}\right)\right\} - \frac{5}{12}\right] \sqrt{\left(\frac{\gamma b}{\nu^*}\right)\left(\frac{APL}{\nu^*}\right)} \qquad 24\ (\cancel{25})$$

where f and g are functions given previously in Eqs. (23) and (24).

Since the value of f is nearly unity and the factor before g in Eq. (24) is small, a good approximate equation for the emissivity is

$$\epsilon \cong \frac{30}{\pi^4}\left(\frac{\theta}{T}\right)^5 e^{-\theta/T}\, \Gamma\left(\frac{3}{4}\right)\left(\frac{2T}{\gamma\theta}\right)^{3/4}\sqrt{\left(\frac{\gamma b}{\nu^*}\right)\left(\frac{APL}{\nu^*}\right)} \qquad 25\ (\cancel{26})$$

IV. APPLICATION TO CARBON MONOXIDE

For carbon monoxide, the molecular constants are

$\theta = 3066.9°K$

$\nu^* = 2142.3\ cm^{-1}$

$\gamma = 0.000895$

$\delta = 0.0091$

$x = 0.00620$

The value of A computed from the measurements of Penner and Weber (Ref. 3) is

$$A = \cancel{21.18}\ 22.95\ atm^{-1}\ cm^{-2}$$

They have also determined (Ref. 8) b to be 0.077 cm^{-1} at one atmosphere of total pressure.

According to the approximate equation (~~26~~ 25), the emissivity at T = 300°K and a total pressure of one atmosphere

$$1.62 \times 10^{-3} \sqrt{pL}$$

$$\varepsilon = \overset{1.597}{\cancel{1.690}} \times 10^{-3} \sqrt{pL} \qquad \overset{26}{(\cancel{27})}$$

where pL is in atm-cm. By using the more exact equation (25), the emissivity is

$$\varepsilon = \underset{1.640}{\overset{1.686}{\cancel{1.640}}} \times 10^{-3} \sqrt{pL} \qquad \overset{27}{(\cancel{28})}$$

The difference between the approximate value and the more exact value is quite small. The comparison between the computed emissivity and the measurements of Ullrich and Hottel (Ref. 4) is shown, in Fig. 1. The agreement is quite satisfactory up to pL of approximately 10.

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Section 3

Servo-Stabilization of Combustion in Rocket Motors

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Servo-Stabilization of Combustion in Rocket Motors

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Summary

This paper shows that the combustion in the rocket motor can be stabilized against any value of time lag in combustion by a feedback servo link from a chamber pressure pickup, through an appropriately designed amplifier and control capacitance, acting on the propellant feed line. The technique of stability analysis is based upon a combination of Satche diagram and Nyquist diagram.

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The phenomenon of rough burning in liquid propellant rocket motor has been interpreted as an instability of the coupled system of propellant feed and the combustion by D. F. Gunder and D. R. Friant (Ref. 1), M. Yachter (Ref. 2), M. Summerfield (Ref. 3), and L. Crocco (Ref. 4). The essential feature of these theories is the time lag between the instant of injection of the propellant and the instant when the propellant is burned into hot gas. Crocco has further improved on this concept by considering the time lag as an integrated effect of consecutive stages, each of which is controlled by the prevailing pressure in the combustion chamber. As a result of this new concept, Crocco shows that there is the possibility of intrinsic instability with constant injection rate not influenced by the chamber pressure.

The present paper will first give a slightly more general formulation of Crocco's concept of time lag, allowing arbitrary pressure dependence of lag. Then the problem of intrinsic stability is discussed by following a method suggested by M. Satche (Ref. 5). This method is based upon a modification of the Nyquist diagram and is particularly useful for systems having time lag. For easy reference, this new diagram will be called the Satche diagram. The later section of the paper will show the possibility of stabilizing the combustion by means of a feed-back servo for all values of time lag. Such possibility of servo-stabilization was first mentioned by W. Bollay in his admirable paper (Ref. 6) on the application of servo-mechanisms to aeronautics. The present study definitely shows the power of this idea.

Time Lag in Combustion

Let $\dot{m}_b(t)$ be the mass rate of generation of hot gas by combustion

at time instant t. Consider, for simplicity, a monopropellant motor. Then the mass rate of injection at t can be denoted by $\dot{m}_i(t)$. Let $\tau(t)$ be the time lag for that parcel of propellant which is burned at the instant t. Then the mass burned during the interval from t to $t+dt$ must be equal to the mass injected during the time from $t-\tau$ to $t-\tau+d(t-\tau)$. Thus

$$\dot{m}_b(t)\,dt = \dot{m}_i(t-\tau)\,d(t-\tau) \tag{1}$$

The mass of hot gas generated is used to fill the combustion chamber by raising its pressure $p(t)$ or it is exhaust through the rocket nozzle. If the frequency of the possible oscillations in the chamber is small, then pressure in the chamber can be considered as uniform and as a first approximation (Ref. 7) the rate of flow through the nozzle can be taken as proportional to the instantaneous chamber pressure $p(t)$. Thus if $\bar{\dot{m}}$ is the steady mass rate flow through the system, $\bar{M}_g$ is the mass of hot gas in the chamber, and if the volume occupied by unburned liquid propellant is neglected,

$$\dot{m}_b\,dt = \bar{\dot{m}}\left(\frac{p}{\bar{p}}\right)dt + d\left(\bar{M}_g\,\frac{p}{\bar{p}}\right) \tag{2}$$

where $\bar{p}$ is the steady state pressure in the combustion chamber.

By following Crocco, the non-dimensional variables are defined as

$$\varphi = \frac{p-\bar{p}}{\bar{p}}, \qquad \mu = \frac{\dot{m}_i-\bar{\dot{m}}}{\bar{\dot{m}}} \tag{3}$$

φ and μ are then the fractional deviation of pressure and injection rate from the average. With Eq. (3), $\dot{m}_b$ can be eliminated from Eqs. (1) and (2), and

$$\frac{\bar{M}_g}{\bar{\dot{m}}}\frac{d\varphi}{dt} + \varphi + 1 = \left(1-\frac{d\tau}{dt}\right)\left[\mu(t-\tau)+1\right] \tag{4}$$

To calculate the quantity $d\tau/dt$, Crocco's concept of pressure dependence of time lag has to be introduced. If the rate at which the liquid propellant is prepared for the final rapid transformation into hot gas is a function $f(p)$, then the lag τ is determined by

$$\int_{t-\tau}^{t} f(p)\,dt = \text{Constant} \tag{5}$$

By differentiating Eq. (5) with respect to t,

$$[f(p)]_t - [f(p)]_{t-\tau}\left(1 - \frac{d\tau}{dt}\right) = 0$$

Now specifically assume that the deviation of the pressure p from the steady state value $\bar{p}$ is small. Then $f(p)$ at the instant t and $f(p)$ at the instant $t-\tau$ can be expanded as Taylor's series around $\bar{p}$. By taking only the first order terms,

$$[f(p)]_t = f(\bar{p}) + \bar{p}\left(\frac{df}{dp}\right)_{p=\bar{p}} \varphi(t)$$

$$[f(p)]_{t-\tau} = f(\bar{p}) + \bar{p}\left(\frac{df}{dp}\right)_{p=\bar{p}} \varphi(t-\tau)$$

Here τ is the lag at the average pressure $\bar{p}$, a constant now. Then

$$1 - \frac{d\tau}{dt} = 1 + \left\{\frac{d\log f}{d\log p}\right\}_{p=\bar{p}} \left[\varphi(t) - \varphi(t-\tau)\right] \tag{6}$$

By combining Eqs. (4) and (6), the following equation is obtained

$$\frac{d\varphi}{dz} + \varphi = \mu(z-\delta) + n\left[\varphi(z) - \varphi(z-\delta)\right] \tag{7}$$

where

$$n = \left[\frac{d\log f}{d\log p}\right]_{p=\bar{p}} \tag{8}$$

and

$$z = t/\theta_g, \qquad \theta_g = \bar{M}_g/\bar{\dot{m}} \tag{9}$$

If n is a constant independent of $\bar{p}$, then $f(p)$ is proportional to p^n. This is the form of $f(p)$ assumed by Crocco. The present formulation of the problem is slightly more general in that $f(p)$ is arbitrary and the value of n is to be computed by using Eq. (8), and is a function of $\bar{p}$. θ_g is, of course, the gas transit time.

Intrinsic Instability

Crocco called the instability of combustion with a constant rate injector the intrinsic instability. If the injector rate is constant not influenced by the chamber pressure p, then $\mu \equiv 0$. Therefore the stability problem is controlled by the following simple equation obtained from Eq. (7),

$$\frac{d\varphi}{dz} + (1-n)\varphi(z) + n\varphi(z-\delta) = 0 \tag{10}$$

Now let

$$\varphi(z) \sim e^{sz}$$

Then

$$s + (1-n) + n e^{-\delta s} = 0 \tag{11}$$

This is the equation for the exponent s.

¶ Crocco determined the value of the complex number s by studying the set of two equations for the real and the imaginary parts of Eq. (11). However if the point of interest is whether the system is stable or not, we can use the well-known Cauchy's theorem with advantage. Let

$$G(s) = e^{-\delta s} - \left[-\frac{1-n}{n} - \frac{s}{n}\right] \tag{12}$$

Then the question of stability is determined by whether $G(s)$ has roots in the right half of the complex s-plane. This question itself can be in turn answered by watching the argument of $G(s)$ when s trace a contour enclosing the right half s-plane. Specifically,

Let s trace clockwisely the contour consisted of the imaginary axis and a large half circle to the right of the imaginary axis (Fig. 1). If the vector $G(s)$ make a number of complete clockwise revolution, then that number is, (according to Cauchy's theorem), the difference between the number of zeros and the number of poles of $G(s)$ in the right half s-plane. Since $G(s)$ evidently has no poles in the s-plane, the number of revolution of $G(s)$ is the number of zeros. Hence for stability, the vector $G(s)$ must not make any complete revolution, as s traces the specified contour. Therefore the stability question can be answered by plotting graphically on the complex plane $G(s)$. This graph is, of course, the well known Nyquist diagram.

A direct application of this method to $G(s)$ given by Eq. (12) is however inconvenient for the complication caused by lag term $e^{-\delta s}$ (Eq. 8). M. Satche (Ref. 5) however proposed a very elegant and ingenious method to treat such system with time lag. Instead of $G(s)$, break it into two parts,

$$G(s) = g_1(s) - g_2(s) \tag{13}$$

where

$$\left.\begin{aligned} g_1(s) &= e^{-\delta s} \\ g_2(s) &= -\frac{1-n}{n} - \frac{s}{n} \end{aligned}\right\} \tag{14}$$

The vector $G(s)$ is thus a vector with vertex on $g_1(s)$ and its tail on $g_2(s)$. The graph of $g_1(s)$ is the unit circle for s on the imaginary axis. For s on the half great circle, $g_1(s)$ is within the unit circle. The graph of $g_2(s)$ is the straight line (Fig. 2) parallel to the imaginary axis when s is on the imaginary axis. When s is on the half great circle, $g_2(s)$ is a half of a great circle closing the contour on the left. A moment's reflection will show that in order for the vector $G(s)$ not to make complete revolutions for any value of δ, the $g_2(s)$ contour

must lie completely out of the $g_1(s)$ contour. That is, for unconditional intrinsic stability,

$$\frac{1-n}{n} > 1 \,, \quad \text{or} \quad \tfrac{1}{2} > n > 0 \,. \tag{15}$$

When $n > \frac{1}{2}$, the $g_1(s)$ contour and the $g_2(s)$ contour intersect. Stability is still possible however, if for $g_2(s)$ within the unit circle (Fig. 3), $g_1(s)$ is to the right of $g_2(s)$. This condition is satisfied if

$$\cos \delta \sqrt{2n-1} > -\frac{1-n}{n}$$

Or if

$$\left.\begin{aligned} &\delta < \delta^* \\ \text{where} \qquad &\delta^* = \frac{\cos^{-1}\left(-\frac{1-n}{n}\right)}{\sqrt{2n-1}} = \frac{1}{\sqrt{2n-1}}\left(\pi - \cos^{-1}\sqrt{\frac{1-n}{n}}\right) \end{aligned}\right\} \tag{16}$$

When $\delta = \delta^*$, then $G(i\omega^*) = 0$, with

$$\omega^* = \sqrt{2n-1} \tag{17}$$

Therefore when $\delta = \delta^*$, φ has the oscillatory solution with the angular frequency ω^*.

These results on intrinsic stability were obtained by Crocco. The present discussion however seems to be simpler with the Satche diagram. For the more complicated stability problem treated below with feed-system characteristics and servo-control, the solution is hardly practical without the Satche diagram.

System Dynamics with Servo-Control

Consider now a system including the propellant feed and a servo-control represented by Fig. 4. In order to approximate the elasticity of the feed line, a spring loaded capacitance is put at the midway point between the propellant pump and the injector. Near the injector there is another capacitance controlled by the servo. The servo receives its signal from the chamber pressure pickup through

an amplifier. If the feed system and the motor design is fixed by the designer, the question is whether it is possible to design an appropriate amplifier so that the whole system will be stable. From the practical point of view, it is desirable to have unconditional stability, i.e., stability for any value of δ, because there is no accurate information on the time lag of combustion.

Let $\dot{m}_0$ be the instantaneous mass flow rate out of the propellant pump and p_0 be the instantaneous pressure at the outlet of pump. The average flow rate must be $\bar{\dot{m}}$. The average pressure is $\bar{p}_0$. The pump characteristics can be represented by the following equation,

$$\frac{p_0 - \bar{p}_0}{\bar{p}_0} = -\alpha \, \frac{\dot{m}_0 - \bar{\dot{m}}}{\bar{\dot{m}}} \qquad (18)$$

If the time rate of change of mass flow is small, α is simply related to the slope of the head-volume curve of the pump at constant speed near the steady-state operating point. For constant pressure pumps or the simple pressure feed, α is zero. For conventional centrifugal pumps, α is approximately 1. For displacement pumps, α is very large.

Let $\dot{m}_1$ be the instantaneous mass rate of flow after the spring loaded capacitance, χ be the spring constant of the capacitance and p_1 the instantaneous pressure at the capacitance. Then

$$\dot{m}_0 - \dot{m}_1 = \rho\chi \, \frac{dp_1}{dt} \qquad (19)$$

where ρ is the density of the propellant, a constant.

In the following calculation, the pressure drop in the line by frictional forces will be neglected. Then the pressure difference $p_0 - p_1$ is due to the acceleration of the flow only. That is

$$p_0 - p_1 = \frac{l}{2A}\frac{d\dot{m}_0}{dt} \tag{20}$$

where A is the cross-sectional area of the feedline, a constant, and l is the total length of the feedline. Similarly, if p_2 is the instantaneous pressure at the control capacitance,

$$p_1 - p_2 = \frac{l}{2A}\frac{d\dot{m}_1}{dt} \tag{21}$$

If the mass capacity of the control capacitance is C, then

$$\dot{m}_1 - \dot{m}_i = \frac{dC}{dt} \tag{22}$$

Since the control capacitance is located very closely to the injector, the inertia of the mass of propellant between the control capacitance and the injector is negligible. Then

$$p_2 - p = \frac{1}{2}\frac{\dot{m}_i^2}{\rho A_i^2} \tag{23}$$

where A_i the effective orifice area of the injector. A_i can be eliminated from the calculation by noting that at steady state, the difference of pressures $\bar{p}_0$ and $\bar{p}$, or $\Delta\bar{p}$ is

$$\bar{p}_0 - \bar{p} = \Delta\bar{p} = \frac{1}{2}\frac{\bar{\dot{m}}^2}{\rho A_i^2} \tag{24}$$

Eqs. (18) to (24) describe the dynamics of the feed system. By a straightforward process of elimination of variables, a relation between $\dot{m}_i$, p and C is obtained. To express this relation in non-dimensional form, the following quantities are introduced, following the notation of Crocco:

$$P = \frac{\bar{p}}{2\Delta\bar{p}}, \quad E = \frac{2\Delta\bar{p}}{\bar{\dot{m}}\,\theta_g}\rho\chi, \quad J = \frac{l\bar{\dot{m}}}{2\Delta\bar{p}A\theta_g} \tag{25}$$

and

$$x = C/\bar{\dot{m}}\theta_g \tag{26}$$

where θ_g the gas transit time given by Eq. (7)

No A'

Then the non-dimensional equation relating φ, μ and x is

$$P\left\{1+\alpha E\left(P+\tfrac{1}{2}\right)\frac{d}{dz}+\frac{JE}{2}\frac{d^2}{dz^2}\right\}\varphi+\left[\left\{1+\alpha\left(P+\tfrac{1}{2}\right)\right\}+\left\{\alpha E\left(P+\tfrac{1}{2}\right)+J\right\}\frac{d}{dz}\right. \qquad (27)$$

$$\left.+\left\{\frac{\alpha EJ}{2}\left(P+\tfrac{1}{2}\right)+\frac{EJ}{2}\right\}\frac{d^2}{dz^2}+\frac{EJ^2}{4}\frac{d^3}{dz^3}\right]\mu+\left[\alpha\left(P+\tfrac{1}{2}\right)\frac{d}{dz}+J\frac{d^2}{dz^2}+\frac{\alpha EJ}{2}\left(P+\tfrac{1}{2}\right)\frac{d^3}{dz^3}+\frac{EJ^2}{4}\frac{d^4}{dz^4}\right]x=0$$

where z is the non-dimensional time variable defined by Eq. (9)

The dynamics of the servo-control is specified by the composite of the instrument characteristics of the pressure pickup, the response of the amplifier and the properties of the servo. Since it is not the purpose of the present paper to discuss the detailed design of the servo-control, the overall dynamics of the servo-control is represented by the following operator equation:

$$F\left(\frac{d}{dz}\right)\varphi = x \qquad (28)$$

where F is the ratio of two polynomials with the denominator of higher order than the numerator.

Eqs. (7), (27) and (28) are the three equations for the three variables φ, μ and x. Since they are equations with constant coefficients, the appropriate forms for the variables are

$$\varphi = a e^{sz}, \quad \mu = b e^{sz}, \quad x = c e^{sz} \qquad (29)$$

By substituting Eq. (29) into Eqs. (7), (27) and (28), three homogeneous equations for a, b and c are obtained. In order for a, b, c to be non-zero, the determinant formed by their coefficients must vanish. This condition can be written as follows:

$$
\begin{aligned}
&[s+(1-n)]\left[\frac{EJ^2}{4}s^3+\frac{EJ}{2}\left\{1+\alpha(P+\tfrac{1}{2})\right\}s^2+\left\{\alpha E(P+\tfrac{1}{2})+J\right\}s+\left\{1+\alpha(P+\tfrac{1}{2})\right\}\right] \\
&+e^{-\delta s}\left\{\frac{nEJ^2}{4}s^3+\left[\frac{nEJ}{2}\left\{1+\alpha(P+\tfrac{1}{2})\right\}+\frac{PJE}{2}\right]s^2+\left[n\left\{\alpha E(P+\tfrac{1}{2})+J\right\}+\alpha EP(P+\tfrac{1}{2})\right]s\right. \\
&\qquad \left.+\left[n\left\{n+\alpha(P+\tfrac{1}{2})\right\}+P\right]+sF(s)\left[\frac{EJ^2}{4}s^3+\frac{\alpha EJ}{2}(P+\tfrac{1}{2})s^2+Js+\alpha(P+\tfrac{1}{2})\right]\right\}=0
\end{aligned}
\tag{30}
$$

This is the equation for determining the exponent s. $F(s)$ is now recognized as the overall transfer function of the servo-control link. The complete system stability depends upon whether Eq. (30) gives roots that have positive real part.

Instability without Servo-Control

The system characteristics without the servo-control can be simply obtained from the basic equation (30) by setting $F(s)=0$. Let it be assumed that the polynomial multiplied into $e^{-\delta s}$ has no root in the positive half s-plane, as is usually the case. Then Eq. (30) can be divided by that polynomial without introducing poles in the positive half s-plane into the resultant function. That is, for the Satche diagram, we have again

$$G(s)=g_1(s)-g_2(s), \qquad g_1(s)=e^{-\delta s}$$

$g_1(s)$ is thus again the "unit circle". $g_2(s)$ is now much more complicated:

$$
g_2(s)=-\left[\frac{s}{n}+\left(\frac{1-n}{n}\right)\right]\frac{\frac{EJ^2}{4}s^3+\frac{EJ}{2}\left\{1+\alpha(P+\tfrac{1}{2})\right\}s^2+\left\{\alpha E(P+\tfrac{1}{2})+J\right\}s+\left\{1+\alpha(P+\tfrac{1}{2})\right\}}{\frac{EJ^2}{4}s^3+\frac{EJ}{2}\left\{1+\alpha(P+\tfrac{1}{2})+\frac{P}{n}\right\}s^2+\left\{\alpha E(P+\tfrac{1}{2})(1+\frac{P}{n})+J\right\}s+\left\{1+\alpha(P+\tfrac{1}{2})+\frac{P}{n}\right\}}
\tag{31}
$$

The intercept of $g_2(s)$ when s is pure imaginary is given by setting $s=0$ in Eq. (31), i.e.,

$$g_2(0) = -\frac{1-n}{n}\,\frac{1+\alpha(P+\frac{1}{2})}{1+\alpha(P+\frac{1}{2})+\frac{P}{n}} \qquad (32)$$

Since all the parameters n, α, P are positive, the magnitude of $g_2(0)$ is now smaller than the magnitude of $g_2(0)$ for the intrinsic stability problem. Thus the effect of the feed-system is to move the $g_2(s)$ curve towards the unit circle of $g_1(s)$ in the Satche diagram. For instance, for $n=\frac{1}{2}$, $g_2(s)$ is just tangent to the unit circle for the intrinsic system without considering the propellant feed. But with propellant feed-system, $g_2(s)$ contour will intersect the unit circle and the system will become unstable for a certain finite value of time lag δ. The influence of the feed-system is thus always de-stabilizing. This is further confirmed by considering the asymptote of $g_2(s)$ for large imaginary s, obtained from Eq. (31). That is

$$g_2(s) \sim -\left[\frac{s}{n} + \left(\frac{1-n}{n} - \frac{2P}{Jn^2}\right) + \cdots\right], \qquad s \gg 1 \qquad (33)$$

Therefore for large imaginary s, $g_2(s)$ approaches asymptotically a line parallel to the imaginary axis at a distance

$$\frac{1-n}{n} - \frac{2P}{Jn^2}$$

to the left of the imaginary axis. The effect of feed-system is again to move $g_2(s)$ towards the unit circle.

It is thus evident that for the parameter n near $\frac{1}{2}$ or larger than $\frac{1}{2}$, it would be impossible to design the system for unconditional stability. In the Satche diagram, $g_1(s)$ contour and $g_2(s)$ will always intersect without a servo-control.

Complete Stability with Servo-Control

If the polynomial $H(s)$,

$$H(s) = \frac{EJ^2}{4}s^3 + \left[\frac{EJ}{2}\left\{1+\alpha(P+\tfrac{1}{2})\right\} + \frac{PEJ}{2n}\right]s^2 + \left[\alpha E(P+\tfrac{1}{2}) + \frac{\alpha EP}{n}(P+\tfrac{1}{2})\right]s + \left[1+\alpha(P+\tfrac{1}{2})+\frac{P}{n}\right]$$
$$+ \frac{1}{n}sF(s)\left[\frac{EJ^2}{4}s^3 + \frac{\alpha EJ}{2}(P+\tfrac{1}{2})s^2 + Js + \alpha(P+\tfrac{1}{2})\right] \qquad (34)$$

which multiplies into $e^{-\delta s}$, has no poles and zeros in the right half s-plane, then the occurrence of zeros of the expression in Eq. (30) in the right half s-plane can be determined from the Satche diagram with

$$g_1(s) = e^{-\delta s}$$

and
$$g_2(s) = -\left[\frac{s}{n} + \frac{1-n}{n}\right]\left[\frac{EJ^2}{4}s^3 + \frac{EJ}{2}\left\{1+\alpha\left(P+\tfrac{1}{2}\right)\right\}s^2 + \left\{\alpha E\left(P+\tfrac{1}{2}\right)+J\right\}s + \left\{1+\alpha\left(P+\tfrac{1}{2}\right)\right\}\right] \Big/ H(s) \tag{35}$$

As s traces the contour of Fig. 1, $g_1(s)$ is again a unit circle. Therefore if simultaneously the $g_2(s)$ contour is completely outside the unit circle, there can be no root of Eq. (30) in the right half s-plane. In other words, if the transfer function F(s) of the servo-control link is so designed as to displace the $g_2(s)$ contour completely out of the unit circle (Fig. 5), then the system is stabilized for all time lags.

As an example, take

$$n = \tfrac{1}{2},\quad P = 3/2,\quad J = 4,\quad E = 1/4,\quad \alpha = 1$$

Then without the servo-control, the $g_2(s)$ is

$$g_2(s) = -\frac{1}{2}\,\frac{(2s+1)(2s^3+3s^2+9s+6)}{s^3+3s^2+6s+6}$$

Of primary interest is the behavior of g_2 when s is a pure imaginary number $i\omega$, ω real. Then

$$g_2(i\omega) = -\frac{1}{2}\,\frac{(6-21\omega^2+4\omega^4)(6-3\omega^2)+\omega^2(21-8\omega^2)(6-\omega^2)}{(6-3\omega^2)^2+\omega^2(6-\omega^2)^2}$$

$$-\frac{1}{2}i\omega\,\frac{(21-8\omega^2)(6-3\omega^2)-(6-21\omega^2+4\omega^4)(6-\omega^2)}{(6-3\omega^2)^2+\omega^2(6-\omega^2)^2}$$

This contour for $\omega > 0$ is plotted in Fig. 6. It is evident that for

sufficiently large values of time lag, the system will be unstable. On the other hand, if the $g_2(s)$ contour can be changed by the servo-control to say

$$g_2(s) = -2\,\frac{(s+2)(s+3)}{(s+6)}$$

Then as plotted in Fig. 6, the new g_2 contour is completely outside of the unit circle of $g_1(s)$. Therefore the system is now unconditionally stable. A straightforward calculation from Eqs. (31) and (35), shows that the required transfer function $F(s)$ for the servo link is

$$F(s) = -4.875\,\frac{(s+1.0528)(s^2+0.7164s+2.6304)}{s(s+2)(s+3)(s+0.5332)(s^2+0.4668s+3.7511)}$$

The servo link has then the character of an integrating circuit. If with given response of the chamber pressure pickup and of the servo for the control capacitance, an amplifier could be designed to give an overall transfer function close to that specified above, the combustion can be stabilized by such a servo-control.

As the second example, take

$$n = \tfrac{1}{2},\quad P = 3/2,\quad J = 4,\quad E = 1/4,\quad \alpha = 0$$

Since $\alpha = 0$, the feed pressure p_0 is thus constant without even variable flow of propellant. The case then corresponds to that of a simple pressure feed. Without the servo-control,

$$g_2(s) = -\frac{1}{2}\,\frac{(2s+1)(2s^3+s^2+8s+2)}{s^3+2s^2+4s+4}$$

When s is pure imaginary,

$$g_2(i\omega) = -\frac{1}{2}\,\frac{(4-2\omega^2)(2-17\omega^2+4\omega^4)+\omega^2(4-\omega^2)(12-4\omega^2)}{(4-2\omega^2)^2+\omega^2(4-\omega^2)^2}$$

$$-\frac{1}{2}i\omega\,\frac{(4-2\omega^2)(12-4\omega^2)-(4-\omega^2)(2-17\omega^2+4\omega^4)}{(4-2\omega^2)^2+\omega^2(4-\omega^2)^2}$$

and yet such possible instability should not be dismissed.

This contour of g_2 is plotted in Fig. 7.... It is evident that without servo-control the combustion will be unstable for sufficiently long time lag. In fact, the system is even less stable than the system considered in the first example: It will become unstable at shorter time lag. The part of the g_2 contour near $\omega = 2$ is of special interest. Near $\omega = 2$, the contour comes so close to the unit circle of g_1 that if the value of time lag δ is such as to make g_1 and g_2 for $\omega \sim 2$ very close to each other, then an almost undamped oscillation at $\omega \sim 2$ can occur. This critical value of δ is evidently smaller than the critical δ determined from the true intersection of g_2 with the unit circle at $\omega \simeq 0.65$. Such near instability at smaller values of time lag can be easily overlooked in the analytic treatment of the stability condition by Crocco. This, perhaps, indicates the superiority the present graphical method.

For unconditional stability, g_2 should be displaced out of the unit circle, to say the same "stable" contour as in the first example. The required transfer function $F(s)$ is calculated to be

$$F(s) = -4.875\,\frac{(s+0.8126)(s^2-0.04337s+2.6506)}{s^2(s+2)(s+3)(s^2+4)}$$

The required servo link must then have the character of double integrating circuit. Furthermore, the transfer function has poles at $\pm 2i$, purely imaginary. This unrealistic requirement comes from the original feed-system dynamics and is due to the neglection of frictional damping in the feed line. In any actual system, the frictional damping in the feed line will remove these pure imaginary poles of the required transfer function $F(s)$ and replace them by two complex conjugate poles.

Stability Criteria

In the preceding discussion of servo-stabilization, it is assumed that the polynomial $H(s)$, Eq. 134) has no pole or zero in the right half s-plane. This is however not necessarily the case. In general then, one should first investigate the number of zeros and poles of $H(s)$ in the right half s-plane. To do this, it should be recognized that the polynomial in Eq. 134) before the factor $F(s)$ usually does not have zeros in the right half s-plane. Therefore instead of studying $H(s)$, one can study the ratio of $H(s)$ and that polynomial. That is, the number of zeros and poles of $H(s)$ in the right half s-plane is the same as the number of zeros and poles of the following expression

$$\frac{H(s)}{\frac{EJ^2}{4}s^3+\left[\frac{EJ}{2}\left\{1+\alpha(P+\tfrac{1}{2})\right\}+\frac{PEJ}{2n}\right]s^2+\left[\alpha E(P+\tfrac{1}{2})+\frac{\alpha EP}{n}(P+\tfrac{1}{2})\right]s+\left[1+\alpha(P+\tfrac{1}{2})+\frac{P}{n}\right]} = 1+K(s) \qquad (36)$$

where

$$K(s) = \frac{\frac{1}{n}sF(s)\left[\frac{EJ^2}{4}s^3+\frac{\alpha EJ}{2}(P+\tfrac{1}{2})s^2+Js+\alpha(P+\tfrac{1}{2})\right]}{\frac{EJ^2}{4}s^3+\left[\frac{EJ}{2}\left\{1+\alpha(P+\tfrac{1}{2})\right\}+\frac{PEJ}{2n}\right]s^2+\left[\alpha E(P+\tfrac{1}{2})+\frac{\alpha EP}{n}(P+\tfrac{1}{2})\right]s+\left[1+\alpha(P+\tfrac{1}{2})+\frac{P}{n}\right]} \qquad (37)$$

According to the Nyquist criterion, the number of poles and zeros for $1+K(s)$ in the right half s-plane can be found by plotting the Nyquist diagram of $1+K(s)$ with s tracing the contour of Fig. 1. In fact, if $1+K(s)$ or $H(s)$ has r zeros and q poles in right half s-plane, then $K(s)$ will carry out $r-q$ clockwise revolutions around the point -1, as s traces the contour of Fig. 1. Hence the first step in the stability analysis is to plot the Nyquist diagram of $K(s)$.

Therefore when one divide the Eq. (30) by $H(s)$ in order to obtain $g_1(s)$ and $g_2(s)$ as given by Eq. (35), q zeros and r poles are introduced in the right half s-plane. The q poles of $K(s)$ must come from $F(s)$, since the polynomial in the denominator of Eq. 137) has no zero in the

right half s-plane. Therefore the original expression in Eq. (30) also has q poles in the right half s-plane. Hence in order for the original expression in Eq. (30) to have no zero in the right half s-plane, $g_2(s)$ must $-q+(q-r)=-r$ clockwise revolutions around the unit circle. In order the stability be unconditional, i.e., stable for all time lag, the $g_2(s)$ contour should never intersect the unit circle. Therefore the general unconditional stability criteria are, first, g_2 contour completely outside of the unit circle, and, second, $g_2(s)$ making r counterclockwise revolutions around the unit circle as s traces the conventional contour enclosing the right half s-plane. These are the criteria for stability with the Satche diagram. To determine r, one has to use the Nyquist diagram of $K(s)$, Eq. (37). Thus the stability problem for the general case requires both the Satche diagram and the Nyquist diagram. (Fig. 8)

Concluding Remarks

In the previous sections of this paper, the theoretical possibility of completely stabilizing the combustion for any value of time lag by servo-control is demonstrated. The great flexibility of electronic amplifier seems to indicate that this theoretical possibility can be realized laboratory. On the other hand, without the servo link, unconditional stability is shown to be generally impossible. Therefore the concept of feed-back servo is indeed a powerful tool in controlling the behavior of a time-lag system. It is to be realized, of course, that the proposed scheme is but one among many. No attempt is made here to give an exhaust treatment of all possible schemes. The best scheme is certainly to be determined by detailed considerations on all aspects of the practical engineering. The main purpose here is to give a general discussion of the concept together with a

suggested general method of analyses by the Satche diagram.

It is of interest to point out that stabilization by servo-control is only one application of the general concept of feed back link. The opposite case of de-stabilization could be of importance also. For instance, in the so-called valveless pulsejet, it is not always possible to operate the engine with the desired pulsation. With a feed back servo linking the combustion chamber pressure pickup through an amplifier to the fuel line, the system can be de-stabilized at the desired operating frequency and thus operate the engine at that frequency of pulsation. This application of servo-destabilization gives the valveless pulsejet a new flexibility and a extended range of operation. Therefore it is recommended that the applications of feed back control to systems with time lag should be carefully explored.

Appendix

Calculation of Parameters J and E

If L^* and c^* are the characteristic length and the characteristic velocity of the motor, and if T_c is the chamber temperature, R the gas constant, the transit time θ_g is

$$\theta_g = \frac{L^* c^*}{R T_c}$$

To calculate J and E defined by Eq. (25), it is more convenient to use the average propellant velocity v in the feed line. Thus

$$\bar{\dot m} = \rho A v$$

Thus

$$J = \frac{1}{2}\rho v\left(\frac{l}{\theta_g}\right)\Big/\Delta\bar p$$

A consistent set of units would be ρ in slugs per cubic foot, v in ft per sec., l in ft, θ_g in sec. and $\Delta\bar p$ in lbs. per square foot.

If d the diameter of the feed line, h its thickness and E' the Young's modulus of the tube material,

$$\chi = l\pi\left(\frac{d}{2}\right)^2 d\big/E'h$$

Therefore

$$E = \frac{2\Delta\bar p}{E'}\left(\frac{d}{h}\right)\frac{l/\theta_g}{v}$$

A consistent set of units would be $\Delta\bar p$ in lb per square inches, E' in lb. per square inches, l in feet, θ_g in sec., and v in ft. per sec.

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Servo-Stabilization of Combustion in Rocket Motors

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Summary

This paper shows that the combustion in the rocket motor can be stabilized against any value of time lag in combustion by a feedback servo link from a chamber pressure pickup, through an appropriately designed amplifier, ~~and~~ to a control capacitance, ~~acting~~ on the propellant feed line. The technique of stability analysis is based upon a combination of the Satche diagram and the Nyquist diagram. For simplicity of calculation, only low frequency oscillations in monopropellant rocket motors are considered. However, the concept of servo-stabilization and method of analysis are believed to be generally applicable to other cases.

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The phenomenon of rough burning in liquid propellant rocket motor has been interpreted as the instability of the coupled system of propellant feed and combustion chamber by D. F. Gunder and D. R. Friant (Ref. 1), M. Yachter (Ref. 2), M. Summerfield (Ref. 3), and L. Crocco (Ref. 4). The essential feature of these theories is the time lag between the instant of injection of the propellant and the instant when the propellant is burned into hot gas. Crocco has further improved on this concept by considering the time lag as an integrated effect of consecutive stages, each of which is controlled by the prevailing pressure in the combustion chamber. As a result of this new concept, Crocco showed the possibility of intrinsic instability with constant injection rate not influenced by the chamber pressure.

The present paper will first give a slightly more general formulation of Crocco's concept of time lag, allowing arbitrary pressure dependence of lag. Then the problem of intrinsic stability is discussed by applying a method suggested by M. Satche (Ref. 5). This method is based upon a modification of the Nyquist diagram and is particularly useful for systems having time lag. For easy reference, this new diagram will be called the Satche diagram. The later sections of the paper will show the possibility of stabilizing the combustion by means of a feedback servo for all values of time lag. Such possibility of servo-stabilisation was first mentioned by W. Bollay in his admirable paper (Ref. 6) on the application of servo-mechanisms to aeronautics. The present study definitely shows the power of this idea.

<u>Time Lag in Combustion</u>

Let $m_b(t)$ be the mass rate of generation of hot gas by combustion at time instant t. Consider, for simplicity, a monopropellant motor. Then the mass rate of injection at t can be denoted by $m_i(t)$. Let (t) be the time lag for that

parcel of propellant which is burned at the instant t. Then the mass burned during the interval from t to t + dt must be equal to the mass injected during the time from t - to t- + d(t-). Thus

(1)

The mass of hot gas generated is either used to fill the combustion chamber by raising its pressure p(t) or is discharged through the rocket nozzle. If the frequency of the possible oscillations in the chamber is small, then the pressure in the chamber can be considered as uniform and as a first approximation (Ref. 7) the rate of flow through the nozzle can be taken as proportional to the instantaneous chamber pressure p(t). Thus if is the steady mass rate flow through the system, is the average mass of hot gas in the chamber, and if the volume occupied by the unburned liquid propellant is neglected,

(2)

where is the steady state pressure in the combustion chamber.

By following Crocco, the non-dimensional variables for the chamber pressure and the rate of injection are defined as

(3)

and are then the fractional deviation of pressure and injection rate from the average. With Eq. (3), can be eliminated from Eqs. (1) and (2), and

(4)

To calculate the quantity , Crocco's concept of pressure dependence of time lag has to be introduced. If the rate at which the liquid propellant is

prepared for the final rapid transformation into hot gas is a function ,
then the lag is determined by

= Constant (5)

By differentiating Eq. (5) with respect to t,

The concept of small perturbation from the steady state will now be explicitly introduced: assume that the deviation of the pressure p from the steady state value is small. Then at the instant t and at the instant t- can be expanded as Taylor's series around . By taking only the first order terms,

Here is the lag at the average pressure , a constant now. Then

(6)

By combining Eqs. (4) and (6), the following equation is obtained

(7)

where

(8)

and

(9)

If n is a constant independent of , then is proportional to .
This is the form of assumed by Crocco. The present formulation of the problem is slightly more general in that is arbitrary and the value of n is to be computed by using Eq. (8), and is a function of . is, of course, the gas transit time.

Intrinsic Instability

Crocco called the instability of combustion with constant rate of injection the intrinsic instability. If the injection rate is constant not influenced by the chamber pressure p, then . Therefore the stability problem is controlled by the following simple equation obtained from Eq. (7),

(10)

Now let

Then (11)

This is the equation for the exponent s.

Crocco determined the value of the complex number s by studying the set of two equations for the real and the imaginary parts of Eq. (11). However if the point of interest is whether the system is stable or not, one can use the well-known Cauchy's theorem with advantage. Let

(12)

Then the question of stability is determined by whether has zeros in the right half of the complex s-plane. This question itself can be in turn answered by watching the argument of when s trace a contour enclosing the right half s-plane. Specifically, let s trace clockwisely the contour consisted of the imaginary axis and a large half circle to the right of the imaginary axis (Fig. 1). If the vector make a number of complete clockwise revolutions, then that number is, according to Cauchy's theorem, the difference between the number of zeros and the number of poles of in the right half s-plane. Since evidently has no poles in the s-plane, the number of revolutions of is the number of zeros. Hence for stability, the vector must not make any complete revolutions, as s traces the specified contour. Therefore the stability question can be answered by plotting graphically on the complex plane. This graph is, of course, the well-known Nyquist diagram.

A direct application of this method to given by Eq. (12) is however inconvenient for the complication caused by lag term (Ref. 8). M. Satche (Ref. 5) however proposed a very elegant and ingenious method to treat such system with time lag: Instead of , break it into two parts,

(13)

where

(14)

The vector is thus a vector with vertex in and its tail on . The graph of is the unit circle for s on the imaginary axis. For s on the large half circle, is within the unit circle. The graph of is

the straight line (Fig. 2) paralled to the imaginary axis when s is on the imaginary axis. When s is on the large half circle, is a half of a great circle closing the contour on the left. A moments' reflection will show that in order for the vector not to make complete revolutions for any value of , the contour must lie completely out of the contour. That is, for unconditional intrinsic stability,

or (15)

When , the contour and the contour intersect. Stability is still possible however, if for within the unit circle (Fig. 3), is to the right of . This condition is satisfied if

Or if

where (16)

When , then with

(17)

. Therefore when , has the oscillatory solution with the angular frequency *.

These results on intrinsic stability were obtained by Crocco. The present discussion with the Satche diagram however seems to be simpler. For the more complicated stability problem treated below with feed-system and servo-control, the solution is hardly practical without the Satche diagram.

<u>System Dynamics with Servo-Control</u>

Consider now a system including the propellant feed and a servo-control represented by Fig. 4. In order to approximate the elasticity of the feed line,

a spring load capacitance is put at the midway point between the propellant pump and the injector. The spring constant is to be computed from the feed line dimensions.* Near the injector there is another capacitance controlled by the servo. The servo receives its signal from the chamber pressure pickup through an amplifier. If the feed system and the motor design is fixed by the designer, the question is whether it is possible to design an appropriate amplifier so that the whole system will be stable. Because there is no accurate information on the time lag of combustion, a practical design should specify unconditional stability, i.e., stability for any value of .

Let be the instantaneous mass flow rate out of the propellant pump and be the instantaneous pressure at the outlet of pump. The average flow rate must be . The average pressure is . The pump characteristics can be represented by the following equation,

(18)

If the time rate of change of mass flow is small, is simply related to the slope of the head-volume curve of the pump at constant speed near the steady-state operating point. For constant pressure pump or the simple pressure feed, is zero. For conventional centrifugal pumps, is approximately 1. For displacement pumps, is very large.

Let be the instantaneous mass rate of flow after the spring loaded capacitance, be the spring constant of the capacitance and the instantaneous pressure at the capacitance. Then

(19)

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* See the Appendix for details.

where is the density of the propellant, a constant.

In the following calculation, the pressure drop in the line by frictional forces will be neglected. Then the pressure difference is due to the acceleration of the flow only. That is

(20)

where A is the cross-sectional area of the feed line, a constant, and is the total length of the feed line. Similarly, if is the instantaneous pressure at the control capacitance,

(21)

If the mass capacity of the control capacitance is C, then

(22)

Since the control capacitance is very close to the injector, the inertia of the mass of propellant between the control capacitance and the injector is negligible. Then

(23)

where A_1 the effective orifice area of the injector. A_1 can be eliminated from the calculation by noting that at steady state, the difference of pressures and , or is

(24)

Equations (18) to (24) describe the dynamics of the feed system. By a straightforward process of elimination of variables, a relation between .

and C is obtained. To express this relation in non-dimensional form, the following quantities are introduced, following the notation of Crocco:

(25)

and (26)

where the gas transit time given by Eq. (9). Then the non-dimensional equation relating , and is

(27)

where is the non-dimensional time variable defined by Eq. (9).

The dynamics of the servo-control is specified by the composite of the instrument characteristics of the pressure pickup, the response of the amplifier and the properties of the servo. Since it is not the purpose of the present paper to discuss the detailed design of the servo-control, the overall dynamics of the servo-control is represented by the following operator equation:

(28)

where F is the ratio of two polynomials with the denominator of higher order than the numerator.

Equations (7), (27) and (28) are the three equations for the three variables , and . Since they are equations with constant coefficients, the appropriate forms for the variables are

(29)

By substituting Eq. (29) into Eqs. (7), (27) and (28), three homogeneous equations for a, b and c are obtained. In order for a, b, c to be non-zero, the determinant formed by their coefficients must vanish. This condition can be written as follows:

(30)

This is the equation for determining the exponent s. F(s) is now recognized as the overall transfer function of the servo-control link. The complete system stability depends upon whether Eq. (3) gives roots that have positive real part.

<u>Instability Without Servo-Control</u>

The system characteristics without the servo-control can be simply obtained from the basic equation (30) by setting F(s) = 0. Let it be assumed that the polynomial multiplied into   has no zero in the positive half s-plane, as is usually the case. Then Eq. (30) can be divided by that polynominal without introducing poles in the positive half s-plane into the resultant function. That is, for the Satche diagram, one has again

is thus again the "unit circle".   is now much more complicated:

(31)

The intercept of when s is pure imaginary is given by setting s = 0 in Eq. (31), i.e.,

(32)

Since all the parameters , , P are positive, the magnitude of is now smaller than the magnitude of given by Eq. (14) for the intrinsic stability problem. Thus the effect of the feed-system is to move the curve towards the unit circle of in the Satche diagram. For instance, for , is just tangent to the unit circle for the intrinsic system without considering the propellant feed. But with the propellant feed-system, contour will intersect the unit circle and the system will become unstable for time lag exceed a certain finite value. The influence of the feed-system is thus always de-stabilizing. This is further confirmed by considering the asymptote of for large imaginary , obtained from Eq. (31). That is

(33)

Therefore for large imaginary s , approaches asymptotically a line parallel to the imaginary axis at a distance

to the left of the imaginary axis. The effect of feed-system is again to move towards the unit circle.

It is thus evident that for the parameter near 1/2 or larger than 1/2, it would be impossible to design the system for unconditional stability. In the Satche diagram, contour and will always intersect without a servo-control.

Complete Stability with Servo-Control

If the polynomial H(s),

(34)

which multiplies into in Eq. (30), has no poles and zeros in the right half s-plane, then the occurance zeros of the expression in Eq. (30) in the right half s-plane can be determined from the Satche diagram with

and

(35)

As s traces the contour of Fig. 1, is again a unit circle. Therefore if simultaneously the contour is completely outside the unit circle, there can be no root of Eq. (30) in the right half s-plane. In other words, if the transfer function F(s) of the servo-control link is so designed as to place the contour completely out of the unit circle (Fig. 5), then the system is stabilized for all time lags.

As an example, take

Then without the servo-control, the is

Of primary interest is the behavior of when s is a pure imaginary number , real. Thus

This contour for is plotted in Fig. 6. It is evident that for sufficiently large values of time lag, the system will be unstable. On the other hand, if the contour can be changed by the servo-control to say

Then as plotted in Fig. 6, the new contour is completely outside of the unit circle of . Therefore the system is now unconditionally stable. A straightforward calculation from Eqs. (31) and (35), shows that the required transfer function F(s) for the servo link is

The servo link has thus the character of an integrating circuit. If with given response of the chamber pressure pickup and of the servo for the control

capacitance, an amplifier could be designed to give an overall transfer function close to that specified above, the combustion can be stabilized by such a servo-control.

As the second example, take

Since , the feed pressure is thus constant with even variable flow of propellant. The case then corresponds to that of a simple pressure feed. Without the servo-control,

When s is pure imaginary,

This contour of is plotted in Fig. 7. It is evident that without servo-control the combustion will be unstable for sufficiently long time lag. In fact, the system is even less stable than the system considered in the first example: It will become unstable at shorter time lag. The part of the contour near 2 is of special interest. Near , the contour comes so close of the unit circle of that if the value of time lag is such as to make and for 2 very close to each other, then an almost undamped oscillation at 2 can occur. This critical value of is evidently smaller than the critical determined from the true intersection of with the unit

circle at 0.65. Such near instability at smaller values of time lag can be easily overlooked in the analytic treatment of the stability condition by Crocco, and yet such possible instability should not be dismissed. This, perhaps, indicates the superiority of the present graphical method.

For unconditional stability, should be displaced out of the unit circle, to, say, the same "stable" contour as in the first example. The required transfer function F(s) is calculated to be

The required servo link must then have the character of double integrating circuit. Furthermore, the transfer function has two purely imaginary poles at . This unrealistic requirement on the amplifier comes from the original feed-system dynamics and is due to the neglection of frictional damping in the feed line. In any actual system, the frictional damping in the feed line will remove these purely imaginary poles of the required transfer function F(s) and replace them by two complex conjugate poles.

Stability Criteria

In the proceding discussion of servo-stabilization, it is assumed that the polynomial H(s), Eq. (34) has no pole or zero in the right half s-plane. This is however not necessarily the case. In general then, one should first investigate the number of zeros and poles of H(s) in the right half s-plane. To do this, it should be recognized that the polynomial in Eq. (34) before the factor F(s) usually does not have zeros in the right half s-plane. Therefore instead of studying H(s), one can study the ratio of H(s) and that polynomial. That is,

the number of zeros and poles of H(s) in the right half s-plane is the same as the number of zeros and poles of the following function

(36)

where

(37)

According to the Nyquist criterion, the number of poles and zeros for 1+K(s) in the right half s-plane can be found by plotting the Nyquist diagram of 1+K(s) with s tracing the contour of Fig. 1. In fact, if 1+K(s) or H(s) has zeros and poles in right half s-plane, then K(s) will carry out clockwise revolutions around the point -1, as s traces the contour of Fig. 1. Hence the necessary information on H(s) can be obtained by plotting the Nyquist diagram of K(s).

When one divides the Eq. (30) by H(s) in order to obtain and as given by Eq. (35), zeros and poles are introduced in the right half s-plane. The poles of K(s) must come from F(s), since the polynomial in the denominator of Eq. (37) has no zero in the right half s-plane. Therefore the original expression in Eq. (30) also has poles in the right half s-plane. Hence in order for the original expression in Eq. (30) to have no zero in the right half s-plane, must make clockwise revolutions around the unit circle. In order the stability be unconditional, i.e., stable for all time lag, the contour should never intersect the unit circle. Therefore the general unconditional stability criteria are, first, contour

completely outside of the unit circle; and, second, making counter-clockwise revolutions around the unit circle as s traces the conventional contour enclosing the right half s-plane. These are the criteria for stability with the Satche diagram. To determine , one has to use the Nyquist diagram of K(s), Eq. (37). Thus the stability problem for the general case requires both the Satche diagram and the Nyquist diagram. (Fig. 8)

Concluding Remarks

In the previous sections of this paper, the theoretical possibility of completely stabilizing the combustion for any value of time lag by servo-control is demonstrated. The great flexibility of electronic amplifier seems to indicate that this theoretical possibility can be always realized. On the other hand, without the servo link, unconditional stability is shown to be generally impossible. Therefore the concept of feedback servo is indeed a powerful tool in controlling the behavior of a time-lag system. It is to be realized, of course, that the proposed scheme is but one among many. No attempt is made here to give an exhaust treatment of all possible schemes. The best scheme is certainly to be determined by detailed considerations on all aspects of the engineering problem, such as the possibility of high frequency accoustic oscillations which are not considered here. The main purpose here is to give a general discussion of the concept together with a suggested general method of analysing the stability by the Satche diagram.

It is of interest to point out that stabilization by servo-control is only one phase of the general concept of feedback link. The opposite case of de-stabilization could be of importance also. For instance, consider the so-called valveless pulsejet. It is not always possible to operate the engine with the desired pulsation. With a feedback servo linking the combustion chamber

pressure pickup through an amplifier to the fuel line, the system can be destabilized at the desired operating frequency and thus operate the engine at that frequency of pulsation. This application of servo-destabilization gives the valveless pulsejet a new flexibility and an extended range of operation. Therefore it seems worthwhile to explore carefully all possible applications of feedback control to systems with time lag.

Appendix

Calculation of Parameters J and E

If L^* and c^* are the characteristic length and the characteristic velocity of the motor, and if T_c is the chamber temperature, R the gas constant, the transit time is

To calculate J and E defined by Eq. (25), it is more convenient to use the average propellant velocity in the feed line. Thus

Thus, according to Eq. (25)

A consistant set of units would be in slugs per cubic foot, in feet per second, in feet, in seconds and in pounds per square foot.

If d the diameter of the feed line, h its thickness and E' the Youngs modulus of the tube material, then , the change in volume of the feed line per unit rise in pressure, is

Therefore Eq. (25) gives

A consistant set of units would be in pounds per square inch, E' in pounds per square inch, in feet, in seconds, and in feet per second.

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Figure Captions

Fig. 1 Contour traced by the variable s for the Satche Diagram or the Nyquist Diagram.

Fig. 2 Stable Satche Diagram for Intrinsic Oscillations; .

Fig. 3 Unstable Satche Diagram for Intrinsic Oscillations; .

Fig. 4 Servo-controlled Liquid Monopropellant Rocket Motor.

Fig. 5 Satche Diagram for the Original and for the Servo-Stabilized System.

Fig. 6 Satche Diagram for the Original and for the Servo-Stabilized System

$$P = 3/2, \quad J = 4, \quad E = 1/4, \quad = 1$$

without servo intersects the unit circle; with servo is outside the unit circle. Numbers beside points are the value of .

Fig. 7 Satche Diagram for the Original and for the Servo-Stabilized System

$$P = 3/2, \quad J = 4, \quad E = 1/4, \quad = 0.$$

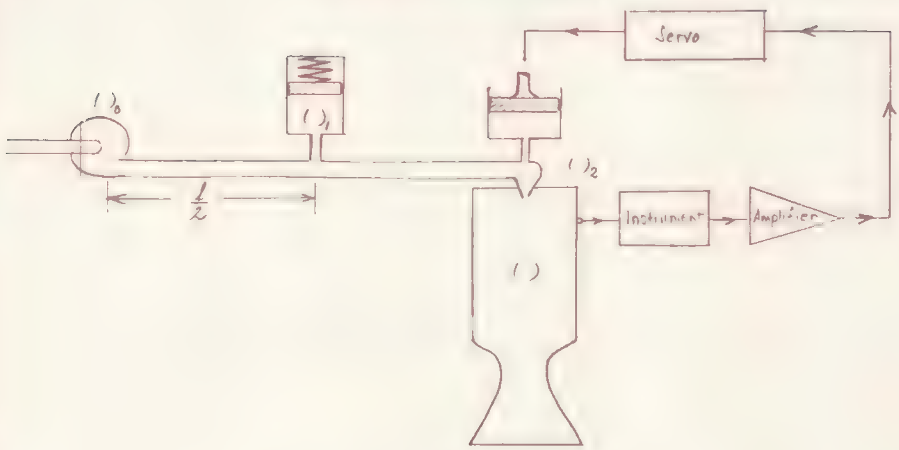
without servo intersects the unit circle; with servo is outside the unit circle. Numbers beside points are the value of .

Fig. 8 Full curve for positive ; dotted curve for negative .
a) Nyquist Diagram for K(s), with two zeros for 1+K(S) in right half s-plane
b) Corresponding stable Satche Diagram

Section 4

Combustion Stabilization by Servo

Combustion Stabilization by Servo



$$\frac{\dot m_0 - \bar{\dot m}}{\bar{\dot m}} = -\alpha \frac{p_0 - \bar p_0}{\bar p_0} \tag{1}$$

$$\dot m_0 - \dot m_1 = \frac{dC_1}{dt}, \quad = \chi \frac{dp_1}{dt}$$

$$p_0 - p_1 = \frac{l}{2A} \frac{d\dot m_0}{dt}$$

$$p_1 - p_2 = \frac{l}{2A} \frac{d\dot m_1}{dt} \qquad \bar p_0 - \bar p = \frac{1}{2} \frac{\bar{\dot m}^2}{\rho A_i^2} = \Delta \bar p$$

$$\dot m_1 - \dot m_i = \frac{dC_2}{dt}$$

$$p_2 - p = \frac{1}{2} \frac{\dot m_i^2}{\rho A_i^2}$$

$$p_0 - p = \frac{l}{2A}\left[\frac{d\dot m_0}{dt} + \frac{d\dot m_1}{dt}\right] + \frac{1}{2}\frac{\dot m_i^2}{\rho A_i^2}$$

$$(p_0 - \bar p_0) - (p - \bar p) = \frac{l}{2A}\left[\frac{d\dot m_0}{dt} + \frac{d\dot m_1}{dt}\right] + 2(\Delta \bar p)\mu$$

$$-\frac{\bar{p}_0}{\bar{p}}\frac{1}{\alpha}\left(\frac{\dot{m}_0-\bar{\dot{m}}}{\bar{\dot{m}}}\right)-\varphi = 2\left(\frac{\Delta\bar{p}}{\bar{p}}\right)\mu + \frac{l}{2A\bar{p}}\left[\frac{d\dot{m}_0}{dt}+\frac{d\dot{m}_1}{dt}\right]$$

$$\dot{m}_0 = \dot{m}_1 + \chi\frac{dp_1}{dt} = \dot{m}_i + \frac{dC_2}{dt} + \chi\left[\frac{d(p_1-p)}{dt}+\frac{dp}{dt}\right]$$

$$p_1 - p = \frac{l}{2A}\frac{d\dot{m}_1}{dt} + \frac{1}{2}\frac{\dot{m}_i^2}{\rho A_i^2}$$

$$= \frac{l}{2A}\frac{d}{dt}\left[\dot{m}_i + \frac{dC_2}{dt}\right] + \frac{1}{2}\frac{\dot{m}_i^2}{\rho A_i^2}$$

$$\frac{d(p_1-p)}{dt} \cong \frac{l}{2A}\frac{d^2}{dt^2}\left[\dot{m}_i + \frac{dC_2}{dt}\right] + \frac{\bar{\dot{m}}}{\rho A_i^2}\frac{d\dot{m}_i}{dt}$$

$$= \frac{l}{2A}\frac{d^2}{dt^2}\left[\bar{\dot{m}}\mu + \frac{dC_2}{dt}\right] + 2(\Delta\bar{p})\frac{d\mu}{dt}$$

$$\frac{dp}{dt} = \bar{p}\frac{d\varphi}{dt}$$

So $$\dot{m}_0/\bar{\dot{m}} = (\mu+1) + \frac{dx_2}{dz} + \chi\left[\frac{l}{2A}\frac{1}{\theta_g^2}\frac{d^2}{dz^2}\left\{\mu+\frac{dx_2}{dz}\right\} + 2\frac{(\Delta\bar{p})}{\bar{\dot{m}}}\frac{1}{\theta_g}\frac{d\mu}{dz} + \frac{\bar{p}}{\bar{\dot{m}}\theta_g}\frac{d\varphi}{dz}\right]$$

$$= (\mu+1) + \frac{dx_2}{dz} + 2\frac{(\Delta\bar{p})}{\bar{\dot{m}}\theta_g}\chi\left[\frac{J}{2}\frac{d^2}{dz^2}\left(\mu+\frac{dx_2}{dz}\right) + \frac{d\mu}{dz} + P\frac{d\varphi}{dz}\right]$$

Put $$2\frac{\Delta\bar{p}}{\bar{\dot{m}}\theta_g}\chi = E$$

$$\left(\frac{\dot{m}_0}{\bar{\dot{m}}}-1\right) = \mu + \frac{dx_2}{dz} + E\left[\frac{J}{2}\frac{d^2}{dz^2}\left(\mu+\frac{dx_2}{dz}\right) + \frac{d\mu}{dz} + P\frac{d\varphi}{dz}\right]$$

$$\dot{m}_1 = \dot{m}_i + \frac{dC_2}{dt}$$

$$-\left(\frac{\Delta\bar{p}}{\bar{p}}+1\right)\frac{1}{\alpha}\left[\mu+\frac{dx_2}{dz}+E\left\{\frac{J}{2}\frac{d^2}{dz^2}\left(\mu+\frac{dx_2}{dz}\right)+\frac{d\mu}{dz}+P\frac{d\varphi}{dz}\right\}\right]-\dot{\varphi}$$

$$=2\left(\frac{\Delta\bar{p}}{\bar{p}}\right)\mu+\frac{\Delta\bar{p}}{\bar{p}}J\left[\frac{d\mu}{dz}+\frac{d^2x_2}{dz^2}+E\left\{\frac{J}{2}\frac{d^3}{dz^3}\left(\mu+\frac{dx_2}{dz}\right)+\frac{d^2\mu}{dz}+P\frac{d^2\varphi}{dz^2}\right\}\right.$$

$$\left.+\frac{d\mu}{dz}+\frac{d^2x_2}{dz^2}\right]$$

$$-\frac{P+\frac{1}{2}}{\alpha}\left[\mu'+\frac{dx_2}{dz}+E\left\{\frac{J}{2}\frac{d^2}{dz^2}\left(\mu'+\frac{dx_2}{dz}\right)+\frac{d\mu'}{dz}+P\frac{d\varphi'}{dz}\right\}\right]-P\varphi'$$

$$=\mu'+J\left[\frac{d\mu'}{dz}+\frac{d^2x_2}{dz^2}+\frac{E}{2}\left\{\frac{J}{2}\frac{d^3}{dz^3}\left(\mu'+\frac{dx_2}{dz}\right)+\frac{d^2\mu'}{dz^2}+P\frac{d^2\varphi'}{dz^2}\right\}\right]$$

$$\boxed{\begin{aligned}&P\left\{1+\frac{P+\frac{1}{2}}{\alpha}E\frac{d}{dz}+\frac{JE}{2}\frac{d^2}{dz^2}\right\}\varphi+\left\{\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+\left(\frac{P+\frac{1}{2}}{\alpha}E+J\right)\frac{d}{dz}+\left(\frac{P+\frac{1}{2}}{\alpha}\frac{EJ}{2}+\frac{EJ}{2}\right)\frac{d^2}{dz^2}\right.\\&\qquad\qquad\left.+\frac{EJ^2}{4}\frac{d^3}{dz^3}\right\}\mu\\&+\left\{\frac{P+\frac{1}{2}}{\alpha}\frac{d}{dz}+J\frac{d^2}{dz^2}+\frac{P+\frac{1}{2}}{\alpha}\frac{EJ}{2}\frac{d^3}{dz^3}+\frac{EJ^2}{4}\frac{d^4}{dz^4}\right\}x_2=0\end{aligned}}$$

$$\begin{vmatrix} p+(1-n)+ne^{-\delta p} & -e^{-\delta p} & 0 \\ P\left[1+\frac{P+\frac{1}{2}}{\alpha}Ep+\frac{JE}{2}p^2\right] & \left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+\left(\frac{P+\frac{1}{2}}{\alpha}E+J\right)p+\frac{EJ}{2}\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)p^2+\frac{EJ^2}{4}p^3 & \left(\frac{P+\frac{1}{2}}{\alpha}p+Jp^2+\frac{P+\frac{1}{2}}{\alpha}\frac{EJ}{2}p^3+\frac{EJ^2}{4}p^4\right) \\ F(p) & 0 & -1 \end{vmatrix} = 0.$$

$$\left[p+(1-n)+ne^{-\delta p}\right]\left[\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+\left(\frac{P+\frac{1}{2}}{\alpha}E+J\right)p+\frac{EJ}{2}\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)p^2+\frac{EJ^2}{4}p^3\right]$$

$$+e^{-\delta p}P\left[1+\frac{P+\frac{1}{2}}{\alpha}Ep+\frac{JE}{2}p^2\right]+pF(p)e^{-\delta p}\left(\frac{P+\frac{1}{2}}{\alpha}+Jp+\frac{P+\frac{1}{2}}{\alpha}\frac{EJ}{2}p^2+\frac{EJ^2}{4}p^3\right)=0.$$

$$\boxed{\begin{aligned} &\left[p+(1-n)\right]\left[\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)p^2+\left(\frac{P+\frac{1}{2}}{\alpha}E+J\right)p+\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)\right] \\ &+e^{-\delta p}\left[n\frac{EJ^2}{4}p^3+\left\{n\frac{EJ}{2}\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+\frac{PJE}{2}\right\}p^2+\left\{n\left(\frac{P+\frac{1}{2}}{\alpha}E+J\right)+\frac{P+\frac{1}{2}}{\alpha}EP\right\}p+\left\{n\left(1+\frac{P+\frac{1}{2}}{\alpha}\right)+P\right\}\right. \\ &\qquad\left.+pF(p)\left\{\frac{EJ^2}{4}p^3+\frac{P+\frac{1}{2}}{\alpha}\frac{EJ}{2}p^2+Jp+\frac{P+\frac{1}{2}}{\alpha}\right\}\right]=0. \end{aligned}}$$

$$e^{-\delta p}\left[1+\frac{pF(p)\left\{\frac{EJ^2}{4}p^3+\frac{P+\frac12}{\alpha}\frac{EJ}{2}p^2+Jp+\frac{P+\frac12}{\alpha}\right\}}{n\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left\{n\left(1+\frac{P+\frac12}{\alpha}\right)+P\right\}p^2+\left\{\left(\frac{P+\frac12}{\alpha}\right)E(n+P)+nJ\right\}p+\left\{n\left(1+\frac{P+\frac12}{\alpha}\right)+P\right\}}\right]$$

$$+\left[p+(1-n)\right]\frac{\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left(1+\frac{P+\frac12}{\alpha}\right)p^2+\left(\frac{P+\frac12}{\alpha}E+J\right)p+\left(1+\frac{P+\frac12}{\alpha}\right)}{n\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left\{n\left(1+\frac{P+\frac12}{\alpha}\right)+P\right\}p^2+\left\{\left(\frac{P+\frac12}{\alpha}\right)E(n+P)+nJ\right\}p+\left\{n\left(1+\frac{P+\frac12}{\alpha}\right)+P\right\}}=0$$

$$\boxed{e^{-\delta p}\left[1-F_1(p)\right]-g_2(p)=0}$$

$$\boxed{F_1(p)=-\frac{pF(p)\left\{\frac{EJ^2}{4}p^3+\frac{P+\frac12}{\alpha}\frac{EJ}{2}p^2+Jp+\frac{P+\frac12}{\alpha}\right\}}{n\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left\{n\left(1+\frac{P+\frac12}{\alpha}\right)+P\right\}p^2+\left\{\left(\frac{P+\frac12}{\alpha}\right)(n+P)E+nJ\right\}p+\left\{n\left(1+\frac{P+\frac12}{\alpha}\right)+P\right\}}}$$

$$\boxed{g_2(p)=-\left[p+(1-n)\right]\frac{\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left(1+\frac{P+\frac12}{\alpha}\right)p^2+\left(\frac{P+\frac12}{\alpha}E+J\right)p+\left(1+\frac{P+\frac12}{\alpha}\right)}{n\frac{EJ^2}{4}p^3+\frac{EJ}{2}\left\{n\left(1+\frac{P+\frac12}{\alpha}\right)+P\right\}p^2+\left\{\left(\frac{P+\frac12}{\alpha}\right)(n+P)E+nJ\right\}p+\left\{n\left(1+\frac{P+\frac12}{\alpha}\right)+P\right\}}}$$

When $p=0$,

$$\boxed{g_2(0)=-\frac{1-n}{n}\,\frac{1+\frac{P+\frac12}{\alpha}}{1+\frac{P+\frac12}{\alpha}+\frac{P}{n}}}$$

When $p\to\infty$

$$g_2(\infty)\cong-\left[p+(1-n)\right]\frac{1}{n}\left[1-\frac{2}{J}\frac{P}{n}\frac{1}{p}\cdots\right]$$

$$=-\frac{1}{n}\left[p+(1-n)-\frac{2}{J}\frac{P}{n}\cdots\right]$$

$$=-\left[\frac{p}{n}+\left(\frac{1-n}{n}-\frac{2}{J}\frac{P}{n^2}\right)\right]$$

$$\boxed{\lim_{\omega \to \infty} \mathcal{R}\, g_2(i\omega) = -\left[\frac{1-n}{n} - \frac{2P}{n^2 J}\right]}$$

$$P = \frac{\bar{p}}{2\Delta\bar{p}}, \qquad J = \frac{\Delta\bar{x}}{2\Delta\bar{p}\, A\, b_1}, \qquad E = \frac{2\Delta\bar{p}}{\bar{r}\, b_1}\frac{dC}{d\bar{p}}.$$

$$(\sim 3/2) \qquad\qquad (\sim 4?) \qquad\qquad (0.25)$$

$$n = 1/2.$$

$$C = \rho\pi r^2 l \quad , \quad \frac{dC}{dp} = 2\pi r l \rho \frac{dr}{dp} = 2\pi r^2 l \rho \frac{d\varepsilon}{dp} = 2\pi r^2 l \rho \frac{1}{E}\frac{d\sigma}{dp}$$

$$= 2\pi r^2 l \rho \frac{1}{E}\frac{R}{t}$$

$$E = \frac{2\Delta\bar{p}}{\cancel{\pi} \cancel{r} v\, \theta_g} 2 \cancel{r} l \cancel{\rho} \frac{1}{E}\frac{R}{t} \qquad\qquad \alpha = \begin{cases} \infty \\ 1 \end{cases}$$

$$= 4\,\frac{\Delta\bar{p}}{E}\,\frac{R}{t}\,\frac{l/\theta_g}{v}$$

$$= 4\,\frac{100}{10\times 10^6}\,\cancel{4}0\,\frac{\cancel{1}0}{\cancel{1}0\times 0.00\cancel{1}}$$

$$= \frac{4}{10\times 1} = \underline{\underline{0.4}}$$

Case 1

$$n = \tfrac{1}{2}, \quad P = 3/2, \quad J = 4, \quad E = 1/4, \quad \alpha = \infty$$

$$g_2(p) = -\frac{(p+\frac{1}{2})(p^3 + \frac{1}{2}p^2 + 4p + 1)}{\frac{1}{2}p^3 + p^2 + 2p + 2} = -\frac{1}{2}\,\frac{(2p+1)(2p^3 + p^2 + 8p + 2)}{p^3 + 2p^2 + 4p + 4}$$

$$g_2(i\omega) = -\frac{1}{2}\,\frac{(1+2i\omega)\{(2-\omega^2) + i\omega(8-2\omega^2)\}}{\{(4-2\omega^2) + i\omega(4-\omega^2)\}}$$

$$= -\frac{1}{2}\,\frac{\{(2-\omega^2) - 2\omega^2(8-2\omega^2)\} + i\omega\{2(2-\omega^2) + (8-2\omega^2)\}}{\{(4-2\omega^2) + i\omega(4-\omega^2)\}}$$

$$= -\frac{1}{2}\,\frac{\{(2-17\omega^2+4\omega^4) + i\omega(12-4\omega^2)\}\{(4-2\omega^2) - i\omega(4-\omega^2)\}}{(4-2\omega^2)^2 + \omega^2(4-\omega^2)^2}$$

$$= -\frac{1}{2}\,\frac{\{(4-2\omega^2)(2-17\omega^2+4\omega^4) + \omega^2(4-\omega^2)(12-4\omega^2)\} + i\omega\{(4-2\omega^2)(12-4\omega^2) - (4-\omega^2)(2-17\omega^2+4\omega^4)\}}{(4-2\omega^2)^2 + \omega^2(4-\omega^2)^2}$$

$$\boxed{g_2(i\omega) = -\frac{1}{2}\,\frac{(4-2\omega^2)(2-17\omega^2+4\omega^4) + \omega^2(4-\omega^2)(12-4\omega^2)}{(4-2\omega^2)^2 + \omega^2(4-\omega^2)^2} - i\frac{\omega}{2}\,\frac{(4-2\omega^2)(12-4\omega^2) - (4-\omega^2)(2-17\omega^2+4\omega^4)}{(4-2\omega^2)^2 + \omega^2(4-\omega^2)^2}}$$

Let $\quad F_1(p) = \dfrac{5/6}{p+1}$

$$1 - F_1(p) = \frac{p + 1/6}{p+1}$$

$$\frac{1}{1-F_1(p)} = \frac{p+1}{p+1/6}$$

$$\frac{1}{1-F_1(i\omega)} = \frac{(1+i\omega)}{1/6 + i\omega} = 6\,\frac{1+i\omega}{1+6i\omega} = 6\,\frac{(1+i\omega)(1-6i\omega)}{1+36\omega^2}$$

$$= \frac{6}{1+36\omega^2}\left[(1+6\omega^2) - 5i\omega\right]$$

ω	ω^2	$4-2\omega^2$	$4-\omega^2$	$2-17\omega^2+4\omega^4$	$12-4\omega^2$	$2[(4-2\omega^2)^2+\omega^2(4-\omega^2)^2]$	[illegible]	[illegible]
0.	0	4	4	2	12	32	−0.2500	0
0.4	0.16	3.68	3.84	−0.6176	11.36	31.803392	−0.1480	−0.5556
0.8	0.64	2.72	3.36	−7.2416	9.44	29.247488	−0.0206	−1.3679
1.2	1.44	1.12	2.56	−14.1856	6.24	21.383168	−0.3328	−2.4302
1.6	2.56	−1.12	1.44	−15.3056	1.76	13.125632	−1.8003	−2.4464
2.0	[illegible]	−4	0	[illegible]	−4	32	[illegible]	[illegible]
2.4	5.76	−7.52	−1.76	36.7904	−11.04	148.785152	+1.1073	−2.3837
2.8	7.84	−11.68	−3.84	114.5824	−19.36	504.055808	+1.4988	−3.7003
3.2	10.24	−16.48	−6.24	247.3504	−28.96	1340.6228	+1.6603	−4.8234
3.6	12.96	−21.92	−8.96	453.5264	−39.84	3041.8718	+1.7473	−5.8427
4.0	16.00	−28	−12	754	−52	6176	+1.8018	−6.8031
4.4	19.36	−34.72	−15.36	1172.1184	−65.44	11546.15	+1.8392	−7.7267
4.8	23.04	−42.08	−19.04	1733.6864	−80.16	20246.45	+1.8664	−8.6255
5.2	27.04	−50.08	−23.04	2461.9664	−96.16	[illegible]	+1.8870	[illegible]
5.6	31.36	−58.72	−27.36	3402.1784	−113.44	53846.36	+1.9031	−10.3748
6.0	36.00	−68	−32	4574	−132.00	82776.	+1.9158	−11.23[illegible]
1.4	1.96	0.08	2.04	−15.9536	4.16	16.326272	−0.9406	−2.8193
1.7	2.89	−1.78	1.11	−13.7216	0.44	13.458338	−1.9177	−1.8250
1.9	3.61	−3.22	0.39	−7.2416	−2.44	21.834962	−0.9106	−0.7294

[illegible]	$1+36u^2$	$1+6u^2$	$\frac{?}{?}\left[\frac{1}{1-F_1}\right]$	$-\frac{?}{?}\left[\frac{1}{1-F_1}\right]$	[illegible]	[illegible]
0	1	1	6	0	− 1.5000	0
[illegible]	6.76	1.96	1.7396	− 1.7751	− 1.2437	− 0.7038
[illegible]	24.04	4.84	1.2080	− 0.9983	− 1.3905	− 1.6319
[illegible]	52.84	9.64	1.0946	− 0.6813	− 2.0200	− 2.4334
[illegible]	93.16	16.36	1.0537	− 0.5152	− 3.1574	− 1.6503
− 1.	145	25	1.0349	− 0.4138	− 0.6725	− 0.9315
[illegible]	208.36	35.56	1.0240	− 0.3456	+ 0.3101	− 2.8236
− 3.[illegible]	283.24	48.04				
[illegible]	369.64	62.44				
[illegible]	467.56	78.76				
[illegible]	577	97				
[illegible]	697.96	117.16				
[illegible]	830.44	139.24				
[illegible]	974.44	163.24				
− 10.37[illegible]	1129.96	189.16				
[illegible]	1297	217				

[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]

Let $\qquad F(p) = -\frac{7}{16}\frac{1}{p^2}$

Then $\qquad F_1(p) = +\frac{7}{16}\dfrac{(p^2+4)}{\frac{1}{2}p^3+p^2+2p+2}$

$$= +\frac{7}{8}\frac{p^2+4}{p^3+2p^2+4p+4}$$

$$1-F_1(p) = \frac{8p^3+9p^2+32p+4}{8(p^3+2p^2+4p+4)}$$

$$g_2^*(p) = g_2(p)/1-F_1(p) = -\frac{1}{2}\frac{(2p+1)(2p^3+p^2+8p+2)}{\cancel{p^3+2p^2+4p+4}}\;\frac{8(p^3+2p^2+4p+4)}{8p^3+9p^2+32p+4}$$

$$= -4\,\frac{(2p+1)(2p^3+p^2+8p+2)}{8p^3+9p^2+32p+4}$$

$$g_2^*(i\omega) = -4\,\frac{(1+2i\omega)\{(2-\omega^2)+i\omega(8-2\omega^2)\}}{(4-9\omega^2)+i\omega(32-8\omega^2)}$$

$$= -4\,\frac{(2-\omega^2)-2\omega^2(8-2\omega^2)+i\omega\{(8-2\omega^2)+2(2-\omega^2)\}}{(4-9\omega^2)+i\omega(32-8\omega^2)}$$

$$= -4\,\frac{\{(2-17\omega^2+4\omega^4)+i\omega(12-4\omega^2)\}\{(4-9\omega^2)-i\omega(32-8\omega^2)\}}{(4-9\omega^2)^2+\omega^2(32-8\omega^2)^2}$$

$$R\,g_2^*(i\omega) = -\frac{(2-17\omega^2+4\omega^4)(4-9\omega^2)+\omega^2(12-4\omega^2)(32-8\omega^2)}{\frac{1}{4}\left[(4-9\omega^2)^2+\omega^2(32-8\omega^2)^2\right]}$$

$$I\,g_2^*(i\omega) = -\omega\,\frac{(12-4\omega^2)(4-9\omega^2)-(32-8\omega^2)(2-17\omega^2+4\omega^4)}{\frac{1}{4}\left[(4-9\omega^2)^2+\omega^2(32-8\omega^2)^2\right]}$$

ω	ω^2	$4-9\omega^2$	$32-8\omega^2$	$2-17\omega^2+4\omega^4$	$12-4\omega^2$	Denominator	$\Re f_2^*$	$\Im f_2^*$
0	0	4	32	2	12	4	-2.0000	0
0.4	0.16	2.56	30.72	-0.6176	11.36	39.3871	-1.3775	-0.4880
0.8	0.64	-1.76	26.88	-7.2416	9.44	116.3799	-1.5049	-1.2239
1.2	1.44	-8.96	20.48	-14.1856	6.24	171.0653	-1.8188	-1.6458
1.4	1.96	-13.64	16.32	-15.9536	4.16	177.0202	-1.9810	-1.6104
1.6	2.56	-19.04	11.52	-15.3056	1.76	175.5651	-1.9555	-1.3015
1.7	2.89	-22.01	8.88	-13.7216	0.44	178.0823	-1.7593	-1.0707
1.9	3.61	-28.47	3.12	-7.2416	-2.44	211.7053	-0.8447	-0.8267
2.0	4.00	-32.00	0	-2.	-4	256.	-0.2500	-1
2.4	5.76	-47.84	-14.08	36.7904	-11.04	857.6412	+1.0082	
2.8	7.84	-66.56	-30.72	114.5824	-19.36			
3.2	10.24	-88.16	-47.92	247.3504	-28.96			
3.6	12.96	-112.64	-71.68	453.5264	-39.84			
4.0	16.00	-140	-96	754	-52			
4.4	19.36	-170.24	-122.88	1172.1184	-65.44			
4.8	23.04	-203.36	-152.32	1733.6864	-80.16			
5.2	27.04	-239.36	-184.32	2466.9664	-96.16			

Let us make

$$g_2^*(p) = \frac{g_2(p)}{1 - F_1(p)} = -K\,\frac{(p+p_1)(p+p_2)}{(p+p_3)}$$

For p large, $\quad g_2^*(p) = -K\left[p + (p_1 + p_2 - p_3) + \cdots\right]$

Now $\quad g_2(p) = -2\,\dfrac{(p+\frac{1}{2})(1+\frac{1}{2}\frac{1}{p}\cdots)}{(1+2\frac{1}{p}\cdots)} = -2\left[p - 1 \cdots\right]$

Thus $\quad K = 2, \quad \boxed{p_1 + p_2 - p_3 = -1}$

Also $\quad \dfrac{K p_1 p_2}{p_3} = 2, \qquad \dfrac{p_1 p_2}{p_3} = 1$

$$\boxed{p_1 p_2 = p_3}$$

Let $p_1 = \frac{1}{2}$, then $p_2 = 2p_3$, or $\quad p_3 = -1 - \frac{1}{2} = -\frac{3}{2}$

No Good

Let $p_1 = \frac{3}{2}$, then $\frac{3}{2}p_2 = p_3$

$$\frac{5}{2} = \frac{1}{2}p_2, \qquad p_2 = 5, \qquad p_3 = \frac{15}{2}$$

$$g_2^*(p) = -2\,\frac{(p+\frac{3}{2})(p+5)}{(p+\frac{15}{2})} = -2\,\frac{(2p+3)(p+5)}{(2p+15)}$$

Let ~~$p_1 = \frac{4}{3}$, $\frac{4}{3}p_2 = p_3$, $\frac{7}{3} = \frac{1}{3}p_2$,~~

Let $p_1 = 2$, $\quad 2p_2 = p_3, \quad p_2 = 3, \quad p_3 = 6$

$$\boxed{g_2^*(p) = -2\,\frac{(p+2)(p+3)}{(p+6)}}$$

Then
$$1 - F_1(p) = \frac{g_2(p)}{g_2^*(p)} = \frac{1}{4}\,\frac{(2p+1)(2p^3+p^2+8p+2)(p+6)}{(p^3+2p^2+4p+4)(p+2)(p+3)}$$

$$F_1(p) = \frac{(p^3+2p^2+4p+4)(p+2)(p+3) - (p+\frac{1}{2})(p+6)(p^3+\frac{1}{2}p^2+4p+1)}{(p^3+2p^2+4p+4)(p+2)(p+3)}$$

$$= \frac{\frac{39}{4}p^3 + \frac{15}{2}p^2 + \frac{51}{2}p + 21}{(p+2)(p+3)(p^3+2p^2+4p+4)}$$

$$= -\frac{p^2 F(p)(p^2+4)}{\frac{1}{2}p^3 + p^2 + 2p + 2}$$

$$= -\frac{2p^2 F(p)(p^2+4)}{p^3+2p^2+4p+4}$$

So
$$\boxed{F(p) = -\frac{\frac{39}{4}p^3 + \frac{15}{2}p^2 + \frac{51}{2}p + 21}{2p^2(p^2+4)(p+2)(p+3)}}$$

$$h(p) = \frac{39}{4}p^3 + \frac{15}{2}p^2 + \frac{51}{2}p + 21$$

$$= 9.75p^3 + 7.5p^2 + 25.5p + 21$$

$$h'(p) = 29.25p^2 + 15p + 25.5$$

$h(-0.8) = 0.408$; $h'(-0.8) = 32.22$

$h(-0.8126) = 0.0005$

$$h(p) = (p+0.8126)(9.75p^2 - 0.4229p + 25.8436)$$

$$\boxed{F(p) = -4.875\,\frac{(p+0.8126)(p^2 - 0.04337p + 2.6506)}{p^2(p+2)(p+3)(p^2+4)}} \leftarrow$$

$$g_2^*(p) = -2\,\frac{(p+2)(p+3)}{(p+6)}$$

$$g_2^*(i\omega) = -2\,\frac{(2+i\omega)(3+i\omega)}{(6+i\omega)} = -2\,\frac{[(6-\omega^2)+5i\omega](6-i\omega)}{(36+\omega^2)}$$

$$= -2\,\frac{[\{6(6-\omega^2)+5\omega^2\}+i\omega\{30-(6-\omega^2)\}]}{(36+\omega^2)}$$

$$= -2\,\frac{36-\omega^2}{36+\omega^2} - 2i\omega\,\frac{24+\omega^2}{36+\omega^2}$$

$$R\,g_2^*(i\omega) = -2\,\frac{36-\omega^2}{36+\omega^2} \quad ; \quad I\,g_2^*(i\omega) = -2\omega\,\frac{24+\omega^2}{36+\omega^2}$$

ω
0
0.4
0.8
1.2
1.4
1.6
1.7
1.9
2.0
2.4
2.8
3.2
3.6
4.0

ω	ω^2	$36+\omega^2$	$\Re\, g_2^*(i\omega)$	$\Im\, g_2^*(i\omega)$			
0	0	36	-2.0000	0			
0.4	0.16	36.16	-1.9823	-0.5345			
0.8	0.64	36.64	-1.9301	-1.0760			
1.2	1.44	37.44	-1.8462	-1.6308			
1.4	1.96	37.96	-1.7935	-1.9149			
1.6	2.56	38.56	-1.7344	-2.2041			
1.7	2.89	38.89	-1.7028	-2.3509			
1.9	3.61	39.61	-1.6354	-2.6488			
2.0	4.00	40	-1.6000	-2.8000			
2.4	5.76	41.76	-1.4483	-3.4207			
2.8	7.84	43.84	-1.2847	-4.0672			
3.2	10.24	46.24	-1.1142	-4.7391			
3.6	12.96	48.96	-0.9412	-5.4353			
4.0	16.00	52	-0.7692	-6.1538			
4.4	19.36	55.36	-0.6011	-6.8925			
4.8	23.04	59.04	-0.4390	-7.6488			
5.2	27.04	63.04	-0.2843	-8.4203			
5.6	31.36	67.36	-0.1378	-9.2048			
6.0	36	72	0	-10.			

Case 2

$$x = \tfrac{1}{2}, \quad P = 3/2, \quad J = 4, \quad E = 1/4, \quad \alpha = 1$$

$$f_2(p) = -(p+\tfrac{1}{2})\,\frac{p^3 + \frac{3}{2}p^2 + \frac{9}{2}p + 3}{\frac{1}{2}p^3 + \frac{3}{2}p^2 + 3p + 3}$$

$$= -\frac{1}{2}\,\frac{(2p+1)(2p^3+3p^2+9p+6)}{p^3+3p^2+6p+6}$$

$$f_2(i\omega) = -\frac{1}{2}\,\frac{(1+2i\omega)\{(6-3\omega^2)+i\omega(9-2\omega^2)\}}{(6-3\omega^2)+i\omega(6-\omega^2)}$$

$$= -\frac{1}{2}\,\frac{\{(6-3\omega^2)-2\omega^2(9-2\omega^2)\} + i\omega\{(9-2\omega^2)+2(6-3\omega^2)\}}{(6-3\omega^2)+i\omega(6-\omega^2)}$$

$$= -\frac{1}{2}\,\frac{\{(6-21\omega^2+4\omega^4)+i\omega(21-8\omega^2)\}\{(6-3\omega^2)-i\omega(6-\omega^2)\}}{(6-3\omega^2)^2+\omega^2(6-\omega^2)^2}$$

$$\mathcal{R}\, f_2(i\omega) = -\frac{1}{2}\,\frac{(6-21\omega^2+4\omega^4)(6-3\omega^2)+\omega^2(21-8\omega^2)(6-\omega^2)}{(6-3\omega^2)^2+\omega^2(6-\omega^2)^2}$$

$$\mathcal{I}\, f_2(i\omega) = -\frac{1}{2}\omega\,\frac{(21-8\omega^2)(6-3\omega^2)-(6-21\omega^2+4\omega^4)(6-\omega^2)}{(6-3\omega^2)^2+\omega^2(6-\omega^2)^2}$$

ω	ω^2	$6-3\omega^2$	$6-\omega^2$	$6-21\omega^2+4\omega^4$	$21-8\omega^2$	[illegible]	$\mathcal{R}\,\varphi_{12}(i\omega)$	$\mathcal{I}\,\varphi_{12}(i\omega)$
0	0	6	6	6	21	72	−0.5000	0
0.4	0.16	5.52	5.84	2.7424	19.72	71.8546	−0.4671	−0.5168
0.8	0.64	4.08	5.36	−5.8016	15.88	70.0666	−0.4396	−1.0948
1.2	1.44	1.68	4.56	−15.9456	9.48	65.5304	−0.5411	−1.6232
1.4	1.96	0.12	4.04	−20.5776	5.32	64.0094	−0.6195	−1.8322
1.6	2.56	−1.68	3.44	−21.5456	0.52	66.2328	−0.6157	−1.7694
1.7	2.89	−2.67	3.11	−21.3024	−2.12	70.1626	−0.5400	−1.7424
1.9	3.61	−4.83	2.39	−17.6816	−7.88	87.8992	−0.1981	−1.7323
2.0	4	−6	2	−14	−11	104	+0.0385	−1.8077
2.2	4.84	−8.52	1.16	−1.9376	−17.72	158.2062	0.5245	[illegible]
2.4	5.76	−11.28	0.24	+17.7504	−25.08	255.1404	0.9207	[illegible]
2.8	7.84	−17.52	−1.84	87.2224	−41.72	666.9870	1.3888	−3.7422
3.2	10.24	−24.72	−4.24	210.3904	−60.92	1590.3380	1.6071	−4.8251
3.6	12.96	−32.88	−6.96	405.6864	−82.68	3399.7548	1.7299	−5.8685
4.0	16	−42	−10	694	−107	6728	1.7878	−6.7779
4.4	19.36	−52.08	−13.36	1098.6784	−133.88	12335.7702	1.8314	[illegible]
4.8	23.04	−63.12	−17.04	1645.5264	−163.32	24348.1314	1.8618	−8.6224
5.2	27.04	−75.12	−21.04	2362.8064	−195.32	35226.2498	1.8842	−9.5045

$$\eta_2^*(p) = -2\,\frac{(p+2)(p+3)}{(p+6)}$$

$$1 - F_1(p) = \frac{1}{4}\,\frac{(2p+1)(2p^3+3p^2+9p+6)}{(p^3+3p^2+6p+6)}\,\frac{(p+6)}{(p+2)(p+3)}$$

$$= \frac{(p+\frac{1}{2})(p+6)(p^3+\frac{3}{2}p^2+\frac{9}{2}p+3)}{(p+2)(p+3)(p^3+3p^2+6p+6)}$$

$$(p+2)(p+3)(p^3+3p^2+6p+6) = (p^2+5p+6)(p^3+3p^2+6p+6)$$

$$\begin{aligned} = {} & p^5+3p^4+6p^3+6p^2 \\ & +5p^4+15p^3+30p^2+30p \\ & \quad 6p^3+18p^2+36p+36 \\ \hline & p^5+8p^4+27p^3+54p^2+66p+36 \end{aligned}$$

$$(p+\tfrac{1}{2})(p+6)(p^3+\tfrac{3}{2}p^2+\tfrac{9}{2}p+3) = (p^2+\tfrac{13}{2}p+3)(p^3+\tfrac{3}{2}p^2+\tfrac{9}{2}p+3)$$

$$\begin{aligned} = {} & p^5+\tfrac{3}{2}p^4+\tfrac{9}{2}p^3+3p^2 \\ & +\tfrac{13}{2}p^4+\tfrac{39}{4}p^3+\tfrac{117}{4}p^2+\tfrac{39}{2}p \\ & \quad 3p^3+\tfrac{9}{2}p^2+\tfrac{27}{2}p+9 \\ \hline & p^5+8p^4+\underbrace{(4+9+3+\tfrac{1}{2}+\tfrac{3}{4})}_{(17+\frac{1}{4})}p^3+(3+29+\tfrac{1}{4}+4+\tfrac{1}{2})p^2 \\ & +\underbrace{(19+\tfrac{1}{2}+13+\tfrac{1}{2})}_{33}p+9 \end{aligned}$$

$32+\frac{3}{4}$

$$F_1(p) = \frac{9\frac{3}{4}p^3+17\frac{1}{4}p^2+33p+27}{(p+2)(p+3)(p^3+3p^2+6p+6)}$$

$$= -\frac{pF(p)\{p^3+p^2+4p+2\}}{\frac{1}{2}p^3+\frac{3}{2}p^2+3p+3} = -\frac{2pF(p)(p^3+p^2+4p+2)}{p^3+3p^2+6p+6}$$

$$\boxed{F(p) = -\frac{9.75p^3+17.25p^2+33p+27}{2p(p^3+p^2+4p+2)(p+2)(p+3)}}$$

$$h_1(p) = p^3 + p^2 + 4p + 2 \qquad h_1(-0.55) = -0.0639$$

$$h_1'(p) = 3p^2 + 2p + 4 \qquad h_1'(-0.55) = 3.8075$$

$$h_1(-0.5332) = 0$$

$$h_1(p) = (p + 0.5332)(p^2 + 0.4668\,p + 3.7511)$$

$$h_2(p) = 9.75\,p^3 + 17.75\,p^2 + 33p + 27$$

$$h_2'(p) = 29.25\,p^2 + 34.5\,p + 33$$

$$h_2(-1.0) = 1.5\,, \qquad h_2'(-1) = 27.75$$

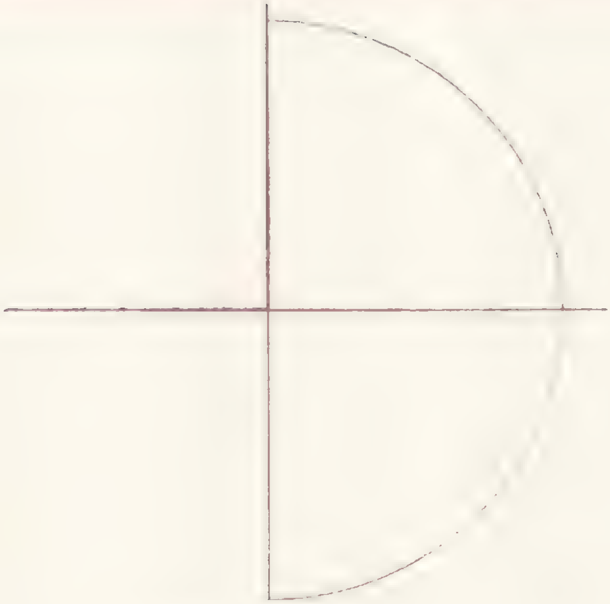
$$h_2(-1.0537) = -0.0264\,, \qquad h_2'(-1.0537) = 29.12$$

$$h_2(-1.0528) = 0$$

$$h_2(p) = (p + 1.0528)(9.75\,p^2 + 6.9852\,p + 25.6460)$$

$$= 9.75\,(p + 1.0528)(p^2 + 0.7164\,p + 2.6304)$$

$$\boxed{F(p) = -\,4.875\,\frac{(p + 1.0528)(p^2 + 0.7164\,p + 2.6304)}{p(p+2)(p+3)(p + 0.5332)(p^2 + 0.4668\,p + 3.7511)}}$$



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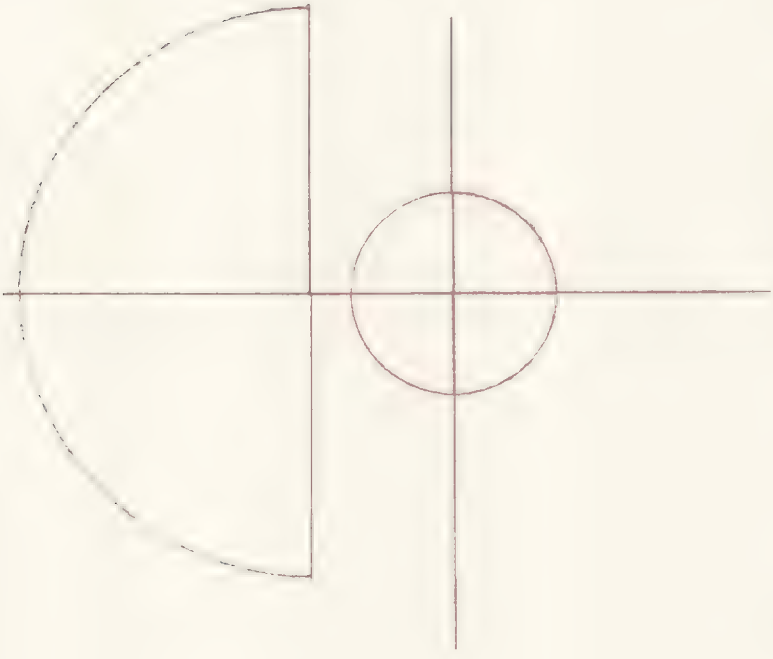

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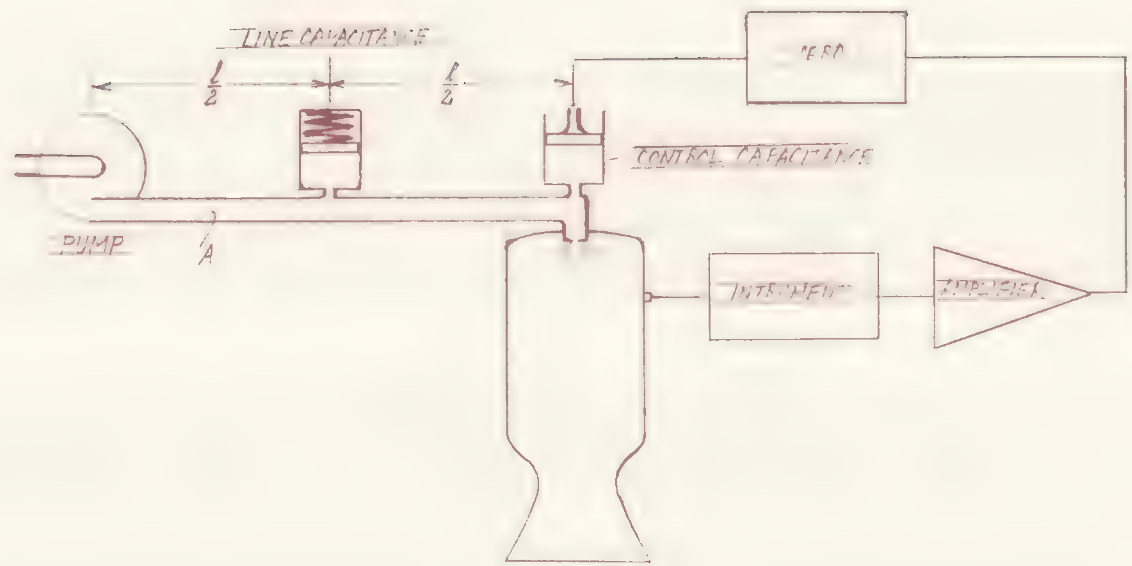
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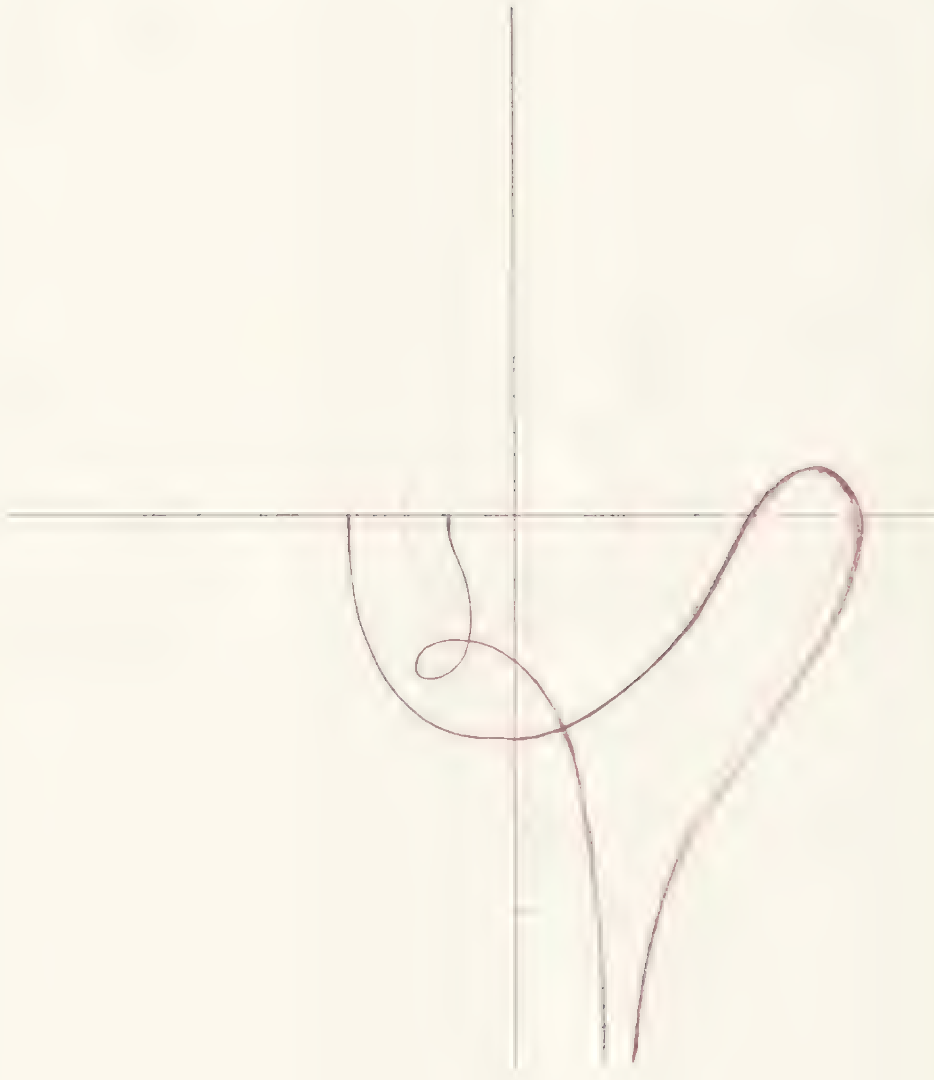

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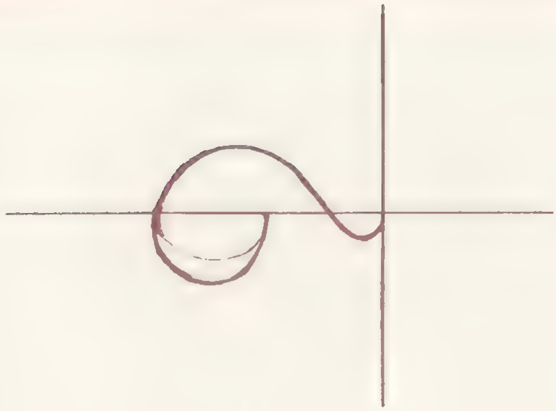

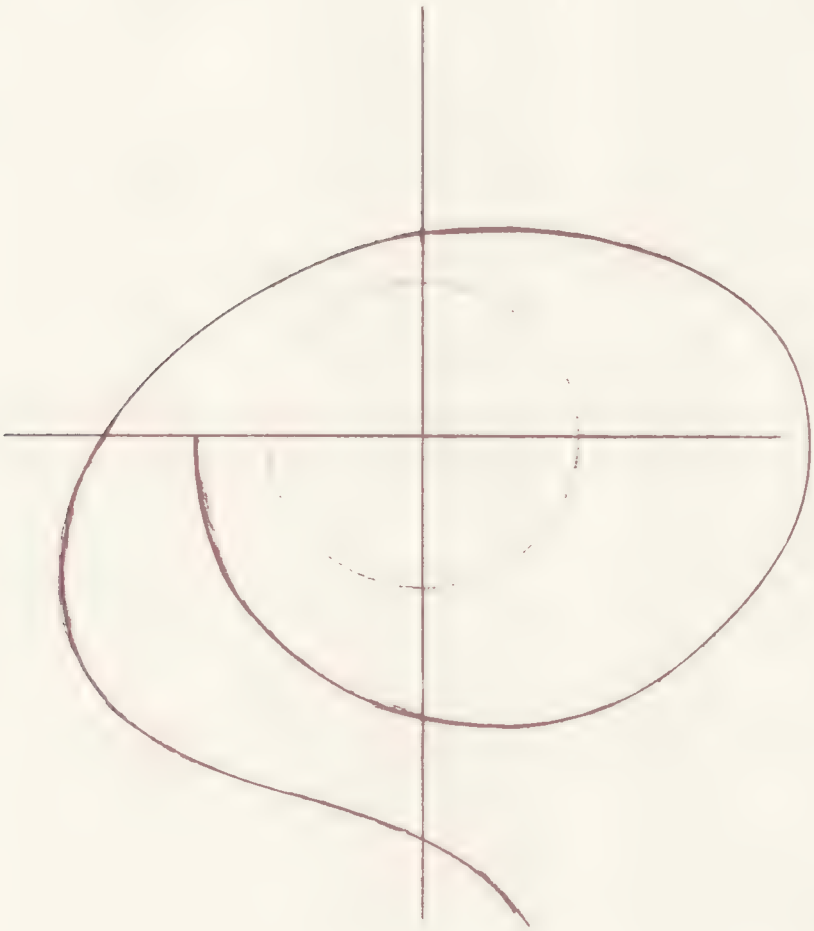

Section 5

Engineering Cybernetics

1.1 Linear Systems of Constant Coefficients

Let us consider the simplest system - a first order system. That is, the differential equation of the system is a first order linear differential equation of constant coefficients. If the system is assumed to be free and is not subjected to "forcing functions", then the differential equation can be written as

$$\frac{dy}{dt} + ky = 0 \tag{1.1}$$

k may be called the spring constant and is real. When there is no variation of y with respect to time, dy/dt vanishes and then Eq. (1.1) requires $y = 0$. Therefore the stationary state or the equilibrium state of the system corresponds to $y = 0$.

The solution of Eq. (1.1) is

$$y = y_0 e^{-kt} \tag{1.2}$$

where y_0 is the initial value of y or

$$y(0) = y_0 \tag{1.3}$$

y_0 is thus the initial disturbance of the system from the equilibrium state. The behavior of the system for $t > 0$, is illustrated in Fig. 1.1 for both positive k and for negative k. It is seen that for $k > 0$, the magnitude of y decreases with time. Then as time increases indefinitely, $y \to 0$. Therefore

Fig. 1.1

for $k>0$, the disturbance of the system will eventually disappear. The system can then be said to be stable. When $k<0$, the disturbed motion of the system increases with time and eventually the disturbance will become very large no matter how small the initial displacement is, and will never return to the equilibrium state once disturbed. Such systems are thus unstable.

For systems of higher order, the differential equation will have higher derivatives. The n-th order system has the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = 0 \qquad (1.4)$$

For a physical system, the coefficients $a_{n-1}, \cdots, a_0$ are real. Then the solution of Eq. (1.4) can be written as

$$y = \sum_{i=1}^{n} y^{(i)} e^{\alpha_i t} \sin(\beta_i t + \varphi_i) \qquad (1.5)$$

where α_i, β_i are real and are related to the coefficients $a_{n-1}, \cdots, a_0$ and φ_i are the phase angles. The motion of the system is thus only stable if all α_i's are negative. If one of them is positive, the disturbance will eventually diverge, and the system is thus unstable.

From the above examples it is seen that the crucial question to ask about the behavior of a linear system of constant coefficients is the question of stability. Needless to say, the usual aim of an engineering design is stability. The question of stability can be answered, however, once the coefficients of the differential equation are specified. In case of the simple first order system specified by Eq. (1.1), the only information that matters is the sign of the coefficient k.

1.2. Linear System with Variable Coefficients

If there is a variable parameter in the system under study, the stationary or the equilibrium state of the system can be changed by changing this parameter. It is natural, then, to expect the coefficients of the linear differential equation describing the system to be also functions of this parameter. For instance, the aerodynamic forces acting on an aircraft are functions of the speed of the aircraft. If the speed of the

Chapter II

Method of Laplace Transform

For linear differential equations with constant coefficients and with time t as the independent variable, the method of Laplace transform is particularly useful in finding the solution. Of course, the problem can be solved by a number of other methods; but the method of Laplace transform appeals specially to the engineering scientists in that it reduces all problems to a uniform basis. The procedure of solution is then standardized and a general approach is possible. The theory and practice of Laplace transform is discussed in many texts.* It is not the purpose of the present chapter to do this. The purpose here is rather to give a summary of results which are useful to our discussion in the subsequent chapters for easy reference. For details and proofs, the reader should consult the texts cited.

2.1 Laplace Transform and Inversion Formula

If $y(t)$ is a function of time variable t defined for $t>0$, then the Laplace transform $Y(s)$ of $y(t)$ is defined as**

$$Y(s)=\int_0^\infty e^{-st}y(t)\,dt \tag{2.1}$$

where s is a complex variable having a positive real part, $\mathcal{R}>0$. For other values of s, the function $Y(s)$ is defined by the analytic continuation. The dimension of $Y(s)$ is the dimension of y multiplied by time.

When $Y(s)$ is known, the original function for which $Y(s)$ is the Laplace transform can be obtained in all cases by the Inversion Formula:

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* See for instance, H. S. Carslaw and J. C. Jaeger. "Operational Methods in Applied Mathematics", Oxford,(1941); or R. V. Churchill, "Modern Operational Methods in Engineering". McGraw Hill. (1944).

For more complete theory, one should consult G. Doetsch, "Theorie und Anwedung der Laplace-Transformation", J. Springer, Berlin. (1937); or D. V. Widder, "The Laplace Transform", Princeton, (1946).

** We shall use throughout capital alphabet to denote the Laplace transform of quantities denoted by a lower case alphabet.

Therefore the error signal vanishes as $t \to \infty$.

Consider now another example of the input: Let the input be sinusoidal,

(48)

$$x(t) = x_{\hat{m}} e^{i\omega t}$$

where $x_{\hat{m}}$ is the amplitude and ω is the frequency. Then

$$X(s) = \frac{x_{\hat{m}}}{s - i\omega} \tag{3.12}$$

The output due to the initial condition is the same as before. The output due to input is given by

$$Y_{\hat{i}}(s) = x_{\hat{m}} \frac{1}{(s - i\omega)(\tau_1 s + 1)} = \frac{x_{\hat{m}}}{1 + i\omega\tau_1}\left[-\frac{1}{s + \frac{1}{\tau_1}} + \frac{1}{s - i\omega}\right]$$

Therefore according to our dictionary, the output $y_{\hat{i}}(t)$ is

$$y_{\hat{i}}(t) = -\frac{x_{\hat{m}}}{1 + i\omega\tau_1} e^{-\frac{t}{\tau_1}} + \frac{x_{\hat{m}}}{1 + i\omega\tau_1} e^{i\omega t}$$

e^{-t/τ_1}

The first term is a pure subsidence and the second term is the steady state output. Thus

use build-up function.

$$[y(t)]_{steady} \Big/ x(t) = \frac{1}{1 + i\omega\tau_1} = F(i\omega)$$

This is in full agreement with our general result given in Eq (2.16). Since

$$\frac{1}{1 + i\omega\tau_1} = \frac{1}{\sqrt{1 + \omega^2\tau_1^2}} e^{-i\tan^{-1}(\omega\tau_1)} \tag{3.13}$$

the steady state output can be expressed as

$$[y(t)]_{steady} = \frac{x_{\hat{m}}}{\sqrt{1 + \omega^2\tau_1^2}} e^{i[\omega t - \tan^{-1}(\omega\tau_1)]}$$

Therefore the amplitude of the steady state output is reduced by the factor $1 \Big/ \sqrt{1 + \omega^2\tau_1^2}$ in comparison with the input, and the phase of the output lags behind the input by the amount $\tan^{-1}(\omega\tau_1)$. For low

the aileron deflection δ . The equation for the roll angle φ is thus

$$I \frac{d^2\varphi}{dt^2} + L_{\dot\varphi} \frac{d\varphi}{dt} = k\delta$$

Now let $p = \frac{d\varphi}{dt}$ be roll speed, then the above equation becomes

$$I \frac{dp}{dt} + L_{\dot\varphi}\, p = k\delta$$

If the roll speed is zero at $t = 0$, then the transformed equation is

$$(Is + L_{\dot\varphi})\, P(s) = k\,\Delta(s)$$

P cf.

The transfer function $F(s)$ is thus

$$\frac{P(s)}{\Delta(s)} = F(s) = \frac{k}{Is + L_{\dot\varphi}} = \frac{k}{L_{\dot\varphi}} \frac{1}{1 + (\frac{I}{L_{\dot\varphi}})s} \tag{3.36}$$

The behavior of the system is determined by the transfer function is thus similar to the cantilever spring with dashpot and the simple lag network. Here the characteristic time τ_1 is $I/L_{\dot\varphi}$. If the damping is very small, then $\tau_1 \to \infty$ and the behavior of the system becomes that of the simple integrator.

3.4 Second Order Systems

Let us return to the cantilever spring with a dashpot, Fig. 3.1. Only now we attach a mass m to the dashpot end. The mass will introduce an inertia force $m\, d^2y/dt^2$, and the equation of motion is now

$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = kx$$

with the initial conditions

$$\left.\begin{aligned} y(0) &= y_0 \\ \left(\frac{dy}{dt}\right)_{t=0} &= y_0^{(1)} \end{aligned}\right\} \tag{3.37}$$

The differential equation of motion can be rewritten in more convenient form by introducing the following parameters:

with $G(s)$ given. The method of Evans determines such roots as functions of the gain K, and is thus called the root-locus method. ital When this is done, any set of specifications on the roots gives a proper choice of the magnitude of K. This method then goes much beyond the mere satisfaction of criterion a) of Section 4.2, but actually solves the design problem for all three criteria stated in that section.

Now let $G(s)$ be specified by its zeros $p_1, p_2, \cdots p_m$ and its poles $q_1, q_2, \cdots q_n$. Then from the definition of gain given by Eqs. (3.16), (3.21) and (3.23),

$$G(s) = A \frac{(s-p_1)(s-p_2)\cdots\cdots(s-p_m)}{(s-q_1)(s-q_2)\cdots\cdots(s-q_n)} \qquad (4.16)$$

where A ~~is a constant real and positive for all physical systems~~

$$A = \frac{(-q_1)(-q_2)\cdots(-q_n)}{(-p_1)(-p_2)\cdots(-p_m)}$$

For physical systems, the polynomials in the numerator and the denominator of $G(s)$ has real coefficients. Then the p's are either real or form complex conjugate pairs. Similarly the q's are either real or form complex conjugate pairs. Therefore A is always real. For engineering systems, usually things are so arranged as to make A not only real but also positive. Hereafter then, we shall consider A to be real and positive. Generally the denominator of $G(s)$ is of equal or higher order than that of the numerator, i.e., $n \geqq m$. Let us express each of the factors in Eq. (4.16) in vector form:

$$\left.\begin{aligned} s - p_1 &= P_1 e^{i\varphi_1} \\ s - p_2 &= P_2 e^{i\varphi_2} \\ &\cdots \\ s - p_m &= P_m e^{i\varphi_m} \end{aligned}\right\} \qquad \text{[illegible]}$$

$$\left.\begin{aligned} s - q_1 &= Q_1 e^{i\theta_1} \\ s - q_2 &= Q_2 e^{i\theta_2} \\ &\cdots \\ s - q_n &= Q_n e^{i\theta_n} \end{aligned}\right\} \qquad \text{[illegible]}$$

The vector $P_r e^{i\varphi_r}$ goes from p_r to s. The vector $Q_r e^{i\theta_r}$ goes from the q_r to s. s is the variable point in the complex s-plane. By using Eqs. (4.18) and (4.19), $G(s)$ can be written as

$$G(s) = A \frac{(P_1 e^{i\varphi_1})(P_2 e^{i\varphi_2})\cdots\cdots(P_m e^{i\varphi_m})}{(Q_1 e^{i\theta_1})(Q_2 e^{i\theta_2})\cdots\cdots(Q_n e^{i\theta_n})} \qquad \text{[illegible]}$$

Since A is real and positive ~~Hence~~, we can write Eq. (4 ... as

$$G(s) = \mathcal{R}\, e^{i\theta} \qquad (4.20)$$

where

$$\mathcal{R} = A\,(P_1 P_2 \cdots P_m / Q_1 Q_2 \cdots Q_n) \qquad (4.21)$$

and

$$\theta = (\varphi_1 + \varphi_2 + \cdots + \varphi_m) - (\theta_1 + \theta_2 + \cdots + \theta_n) \qquad (4.22)$$

Since P's and Q's are magnitudes of vectors defined by Eqs. (4.18) and (4.19), they are positive. Therefore $\mathcal{R}$ is positive. The basic equation for the roots of inverse system transfer function, Eq. (4.15), is thus

$$\frac{1}{K\mathcal{R}}\, e^{-i\theta} = -1$$

Therefore to satisfy this equation, we must have

$$K\mathcal{R} = 1 \qquad (4.23)$$

and

$$\theta = \pm\pi \qquad (4.24)$$

The method of Evans consists of two steps: The first step is to determine all s that satisfy the appropriate angle condition of Eq. (4.24). Then knowing such root locus, we can compute $\mathcal{R}$ and hence K, by Eq. (4.23), for each point on the root-locus. Evans has developed a number of useful rules for plotting the root-locus. We shall explain these rules presently.

Rule 1 For $K = 0$, Eq. (4.15) shows that $G(s) \to \infty$. Thus for $K = 0$, the roots of $1/F_s(s)$ are poles of $G(s)$, or the root-locus starts at the

oscillating servomechanisms are less flexible than are oscillating control servomechnisms in which the oscillation is supplied by an independent generator.

An elementary precaution to be observed, in order that the curve, which is constrained to pass through the point -2, shall avoid the neighborhood of the point -1, is that the curve should interest the real axis, at the point -2, perpendicularly. This implies that the vector $1/F(i\omega)$ should be varying slowly in amplitude, and rapidly in angle, at the frequency at which the system oscillates.

6.6 General Oscillating Control Servomechanism

A relay ~~or a limiter~~ is a non-linear device. But by mixing the signal with a sinusoidal oscillation of high frequency and large amplitude, the output is made to be linear with respect to the signal. Thus the essential concept of oscillating control servomechanisms is the linearization of a non-linear system. J. M. Loeb* has shown that this concept is applicable to any non-linear system, and he calls this method the general linearizing process for non-linear control systems. We shall call the resulting servomechanism the general oscillating control servomechanism.

Let us consider a general function $y(x)$ where y is the output and x is the input. If instead of the variable x, we substitute the sum $x+\epsilon$ where ϵ is much smaller than x. Then if the function $y(x)$ is regular, we can expand $y(x+\epsilon)$ into a Taylor series as

$$y(x+\epsilon) = y(x) + \epsilon\left(\frac{dy}{dx}\right)_x + \epsilon^2 \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_x + \cdots \tag{6.15}$$

We now specify the input x as a periodic function of time t with the period T, and ϵ as a constant. Then it is clear that $y(x)$ is also a periodic function of time with the same period T. Same is true for dy/dx and d^2y/dx^2. Periodic functions can be expanded into Fourier series; thus if we neglect powers of ϵ higher than first, we have

$$\begin{aligned} y(x+\epsilon) \cong{} & a_{00} + \sum_{n=1}^{\infty}(a_{0n}\cos n\omega t + b_{0n}\sin n\omega t) \\ & + \epsilon\left[a_{10} + \sum_{n=1}^{\infty}(a_{1n}\cos n\omega t + b_{1n}\sin n\omega t)\right] \end{aligned} \tag{6.16}$$

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* J. M. Loeb, Annales des Télécommunications, 5:65-71 (1950).

$$F_2^*(s) = \frac{t_0}{2\pi i} \sum_{n=0}^{\infty} e^{-nt_0 s} \int_{\gamma-i\infty}^{\gamma+i\infty} F_2(q)\, e^{nt_0 q}\, dq$$

$$= \frac{t_0}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F_2(q)\, dq \sum_{n=0}^{\infty} e^{-nt_0(s-q)} = \frac{t_0}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F_2(q)\, dq}{1 - e^{-t_0(s-q)}} \tag{7.11}$$

We proceed to evaluate the right-hand member of (7.11) by the method of residues.

The integrand has certain poles, the poles of $F_2(s)$, lying to the left of the path of integration, and other poles, which are the roots of the equation $1 - e^{-t_0(s-q)} = 0$, lying to the right of the path of integration. It is easily seen that the integration upward along the line $\gamma - i\infty$ to $\gamma + i\infty$ is equivalent to integration in the clockwise direction along the closed contour formed by that line and the infinite semicircle in the right half-plane. Hence the right-hand member of Eq. (7.11) is $-t_0$ times the sum of the residues of the integrand with respect to the several roots of the equation $1 - e^{-t_0(s-q)} = 0$.

Now the typical root of the equation is $q = s + 2\pi i m/t_0$, where m is an integer, and the residue of the intagrand with respect to that pole is $-\frac{1}{t_0} F_2(s + 2\pi i m/t_0)$. Therefore finally

$$F_2^*(s) = \sum_{m=-\infty}^{m=\infty} F_2(s + 2\pi i m/t_0) \quad , \quad \mathrm{Re}\, s > \gamma \tag{7.12}$$

This formula gives considerable insight into the properties of $F_2^*(s)$ and at times may be useful in making approximate calculations. However, we can easily obtain an exact representation of $F_2^*(s)$ in finite form.

The function $F_2(s)$ can be represented as the sum of a finite number of partial fractions, thus:

$$F_2(s) = \sum_{k=1}^{n} \frac{a_k}{s - s_k} \tag{7.13}$$

Chapter VIII

Linear Systems with Time Lag

In this chapter, we shall introduce another new element into our linear systems with constant coefficient: the time lag. By time lag, τ (43) we mean that the relation between the different variables of the system cannot be expressed as a relation of these variables all taken at some time instant t ; but on the contrary, the relation involves variables, some taken at the time instant t , and some taken at an earlier instant $t-\tau$. Those taken at the instant $t-\tau$ then lag by the interval τ behind the variables taken at the instant t . This time lag is thus quite different from the characteristic time constant of a first order linear system introduced in Section 3.1. Time lag systems are represented by differential-difference equations of constant coefficients and are more complex than the linear systems studied previously which are represented by differential equations. Systems with time lag were studied by many investigators; for instance, A. Callander, D. Hartree, and A. Porter* and N. Minorsky.** Our interest here is, however, somewhat more restricted. We wish to know: How can we analyse the performance of a feedback servomechanism if there is a characteristic time lag τ in the system? We wish, specifically, to modify the method of Nyquist of Section 4.3, to time lag systems.

We shall develop the theory by treating a particular example of such systems; the example of stabilizing the combustion in a rocket motor by feedback control. The problem of combustion instability in rocket motors was treated by many authors, the following analysis of combustion lag time originates from the work of L. Crocco.*** For simplicity of calculation,**** we shall consider only the case of so called low frequency oscillation in a rocket motor using single liquid propellant.

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* A. Callander, D. Hartree and A. Porter, Phil. Trans. Royal Society of London (A), 235:415-444 (1935).

** N. Minorsky, J. Appl. Mechanics (ASME) 9:67-71 (1942).

*** L. Crocco, J. American Rocket Society, 21: 163-178 (1951).

**** The following discussion is based upon a paper in J. American Rocket Society, 22:256-262 (1952).

Chapter IX

Linear Systems with Stationary Random Inputs

In the previous chapters, the inputs to a system are considered to be definitely specified functions of time t. However, there are many engineering problems for linear systems with constant coefficients where the inputs cannot be so definitely described. An example of such engineering problem is the problem of the motion and the stresses induced in the structure of an airplane wing in turbulent air stream. Here the input can be considered to be the time varying air-flow pattern. But the airflow pattern cannot be described as a definite function of time, but has to be recognized as a random function of time, specified by certain statistical characteristics. It is then evident, the output of the system, the stresses in this case, must be also a random function and can be described also only in statistical terms. The first objective of this chapter is then to find a convenient method of calculating the statistical properties of the output from the specified statistical properties of the input. This forms an easy extention of the early investigations by P. Langevin of the Brownian motion.

Another example of random input is the so-called *noise* in control signals. The noise is introduced by the disturbances and the fluctuations beyond the control of the designer. The problem of noise is a problem of much research in connection with communications engineering. There the central question is how to ~~device a~~ design the system ~~of code~~ so that the effects of the unavoidable noise can be minimized and the useful information of the signal is not destroyed. We shall discuss this particular problem of noise filtering in Chapter 16. The problem of ~~the following discussion in~~ this chapter is, however, somewhat different: In our problem, the random output is the only output of the system. Our purpose in the design of the system, particularly the design of the feedback servomechanism, is to obtain with a given input an output of the desired statistical characteristics. We shall see that the method of transfer function developed in the previous chapters remain useful in the present task.

9.1. Statistical Description of a Random Function

Let us consider a system which generates a random function $y_1(t)$. Now to formulate the concept of a statistical description of such a random

Chapter XII

Linear System with Variable Coefficients

The only system with time varying coefficients considered in detail in the previous chapters is the pendulum with a periodic force at the supporting end, discussed in connection with the phenomenon of parametric excitation and damping. All other systems considered do not have coefficients of their differential equations that are explicitly functions of time. We have, however, shown in Chapter 1, that linear systems with time varying coefficients can have behavior entirely different from systems with constant coefficients. In this chapter, we shall again take up this question and discuss in some detail such a typical but simple system: the short range artillery rocket. We shall demonstrate that the question of stability of such a system with variable coefficients cannot be solved in the same manner as the linear system with constant coefficients. Not only is the method of Laplace transform and transfer function useless for the purpose, but also we are forced to change our entire approach to the problem.

We shall study the motion of a fin-stabilized artillery rocket during the period of action of the rocket thrust. We shall be particularly concerned about the angular deviations of the rocket axis from the launching angle due to the action of disturbances when leaving the launcher and the subsequent damping action of the fins. The general problem of dynamics of artillery rockets has been studied in great detail by various authors in different countries during World War II. The American work is summarized by Rosser, Newton, and Gross.* The work done in England is reported by Rankin.** Carriere's paper*** represents French investigation on the same subject. Our discussion here will be greatly simplified and has the purpose of only bringing out

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* J. B. Rosser, R. R. Newton, and G. L. Gross, "Mathematical Theory of Rocket Flight", McGraw Hill, New York (1947).

** R. A. Rankin, Philosphical Transcations, Roy. Soc. of London, (A), 241:457-585, (1949).

*** P. Carriére, Mémorial de l'Artillerie Française, 25:253-360 (1951).

automatic sensing and measuring control system, i.e., an optimalizing system which automatically holds the airplane at the measured optimum operating conditions.

Of course, a skilled human operator controls the performance of a machine on the optimalizing principle: He watches the instrument readings of the inputs and outputs of the machine, and then uses his knowledge and experience to decide in what directions should the controls be adjusted. The adjusted inputs bring new output readings which have to be interpreted by the operator to determine whether the optimum operating condition is reached or exceeded. New adjustments of the control will have to be made. The continuous adjustment of inputs is the sensing process and the reading of the outputs is the feedback. However, manually-controlled optimalizing systems are necessarily slow in response, and for complicated systems human skill, no matter how developed, is not sufficient. Automatic optimalizing control was conceived by C. S. Draper, Y. T. Li and H. Laning, Jr.* Its application to cruise control of airplane was discussed by J. R. Shull.**

15.2 Principles of Optimalizing Control

The heart of an optimalizing control system is the non-linear component which characterizes the optimum operating conditions. For simplicity of discussion, we shall assume that this basic component has a single input and a single output. For the time being, we shall neglect also any time effects and assume that the output is determined only by the instantaneous value of ~~the~~ input. Since there is an optimum operating point, output as a function of input has a maximum at y_0 and x_0, as shown in Fig. 15.1. It is convenient to refer the output and the input to the optimum point and put the physical input as $x + x_0$, and the physical output as $y^* + y_0$. The optimum point is then the point $x = y^* = 0$. The purpose of an optimalizing control is then to search out this optimum point and to keep the

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* Y. T. Li, Instruments, 25: 72-77, 190-193, 228, 324-327, 350-352 (1952). C. S. Draper and Y. T. Li, "Principles of Optimalizing Control Systems and an Application to Internal Combustion Engine", ASME Publications (1951).

** J. R. Shull, Trans. I.R.E. (Electronic Computers), Dec. 1952, pp. 47-51.

optimum filter design ~~by~~ if we abandon~~ing~~ the inadequate RC-circuits and ~~by~~ actually ~~using~~ use an analog computer or even a digital computer to serve as the filter. Then the theoretical optimum performance can be actually attained. However, the introduction of electro mechanical computer as a component of the filtering system certainly greatly increases the complexity of the overall system, and can be justified only in very critical cases. But if we have made the system very complicated at ~~the~~ high cost, we may ask whether we will actually obtain the very best performance. The optimum performance in the theory discussed in the previous sections is only optimum within the limitations of the assumptions of the theory. For instance, two random signals with the same correlation function or the same power spectrum, according to the theory developed, require the same optimum filter. This is, in a sense, a certain looseness in design criteria. Surely, if we have more statistical information about the signal than just power spectrum, we should be able to distinguish these two signals and to improve our design by utilizing such additional knowledge. Then we can obtain even better performance than possible with the so called "optimum" filter. It is evident that this generalized approach to the filtering problem must require more advanced theory of probability than we have used. The recently developed science of information theory may also find important applications here. A beginning* has been made in this "probabilistic" approach to problem of detecting signal in noise. But much remains to be done. ital.

The Wiener-Kolmogoroff theory of optimum filter is based upon the mean square error criterion. By using this criterion, we essentially put the emphasis on minimizing the large errors without much consideration on the small errors. However in many occasions, we may be most interested in making the frequent error as small as possible, while not particular about making an infrequent large error. It is also possible that the probability function is very lopsided, with the mean far from the mode. For such cases, the mean square error criterion is entirely

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* See for instance, P. M. Woodward, I. L. Davies, Phil. Mag. 41:1001-1017 (1950); Proc. I.R.E. 39:1521-1524 (1951); J.I.E.E. (London) 99(III); 37-51 (1952). T. G. Slattery, Proc. I.R.E. 40:1232-1236 (1952).

Chapter XVIII

Control of Error

In the preceding chapter, we have shown how the principle of ultrastability can make the control system insensitive to accidental errors and occasional failures of the components by the simple device of changing the characteristics of the system whenever instability occurs. Since an ultrastable system will automatically seek stability, the control system, when designed, actually embodies unstable fields of behavior as well as stable fields of behavior. In other words, during the design of an ultrastable system, we make no attempt to predict stability from instability, to separate the right fields of behavior from the wrong fields of behavior. Errors of behavior are merely treated as a probability, but otherwise unspecified. In this chapter, we shall approach the reliability of complex control system from a different point of view: We shall specifically introduce errors into the system and ask how should the system be designed so that the system will give satisfactory performance in spite of the errors. That is, we wish to know how to control the error.

This subject of control of error is now in its early period of development. Only the control of error in most elementary operation can be discussed, and this is wholly due to J. von Neumann.* Our discussion in this chapter is then an exposition of Neumann's work. Its purpose is to serve as an introduction to this very important topic and to indicate the need for much further investigations.

18.1 Reliability by Duplication

It is common knowledge that reliability of a system can generally be increased by the simple expedient of duplication. For instance, if a simple system as shown in Fig. 18.1a has the characteristic that when it fails to operate, it merely gives no output. Then to guard against probability of failure, we can duplicate the system with n identical

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* J. von Neumann, "Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components". Printed notes of lectures given at the California Institute of Technology, Pasadena, California, (1952). California Institute of Technology, 1½ space

$$\vartheta = (1-\mu^2) + 2\varepsilon(\mu^2 - \tfrac{1}{2}) + \sqrt{\frac{(1-2\varepsilon)^2\mu^2(1-\mu)^2 + \varepsilon(1-\varepsilon)}{n}}\; y \qquad (18.28)$$

Eqs. (18.27) and (18.28) have a first term identical with Eq. (18.1). The additional terms come from the imperfect elements and from the statistical distribution of errors.

With any specified ξ, η, ε and n, Eqs. (18.26), (18.27), and (18.28) enable us to compute the distribution function of ϑ, the fraction of activated output lines of the complete Scheffer stroke system. We can make this somewhat clearer by reverting to the notation of probability distribution functions. Thus for instance Eq. (18.26) is equivalent to

$$W(\zeta;\xi,\eta;n) = \frac{e^{-\frac{1}{2}\left\{\frac{\zeta - \left[(1-\xi\eta) + 2\varepsilon(\xi\eta - \frac{1}{2})\right]}{\sqrt{\frac{(1-2\varepsilon)^2\xi(1-\xi)\eta(1-\eta)+\varepsilon(1-\varepsilon)}{n}}}\right\}^2}}{\sqrt{2\pi\,\frac{(1-2\varepsilon)^2\xi(1-\xi)\eta(1-\eta)+\varepsilon(1-\varepsilon)}{n}}}$$

The probability distribution function of ϑ, $W(\vartheta;\xi,\eta;n)$, is thus the result of integrating with respect to ζ and μ of the joint probability of ζ, μ, and ϑ. Thus

$$W(\vartheta;\xi,\eta;n) = \frac{1}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{\frac{(1-2\varepsilon)^2\xi(1-\xi)\eta(1-\eta)+\varepsilon(1-\varepsilon)}{n}}}\cdot$$

$$\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} d\zeta\, \frac{\exp\left(-\frac{1}{2}\left[\left\{\frac{\zeta-\left[(1-\xi\eta)+2\varepsilon(\xi\eta-\frac{1}{2})\right]}{\sqrt{\frac{(1-2\varepsilon)^2\xi(1-\xi)\eta(1-\eta)+\varepsilon(1-\varepsilon)}{n}}}\right\}^2 + \left\{\frac{\mu-\left[(1-\zeta^2)+2\varepsilon(\zeta^2-\frac{1}{2})\right]}{\sqrt{\frac{(1-2\varepsilon)^2\zeta^2(1-\zeta^2)+\varepsilon(1-\varepsilon)}{n}}}\right\}^2 + \left\{\frac{\vartheta-\left[(1-\mu^2)+2\varepsilon(\mu^2-\frac{1}{2})\right]}{\sqrt{\frac{(1-2\varepsilon)^2\mu^2(1-\mu)^2+\varepsilon(1-\varepsilon)}{n}}}\right\}^2\right]\right)}{\sqrt{\frac{(1-2\varepsilon)^2\zeta^2(1-\zeta^2)+\varepsilon(1-\varepsilon)}{n}\cdot\frac{(1-2\varepsilon)^2\mu^2(1-\mu^2)+\varepsilon(1-\varepsilon)}{n}}} \qquad (18.29)$$

We shall now show that under proper conditions we can obtain almost perfect performance of the multiplexed Scheffer stroke system by increasing n. Consider a given fiduciary level δ. Perfect performance means the implication of $\vartheta \leq \delta$, or non-activation of output, by $\xi \geq 1-\delta$, $\eta \geq 1-\delta$, or the activation of both inputs; the implication of $\vartheta \geq 1-\delta$, by either $\xi \leq \delta$, $\eta \geq 1-\delta$ or $\xi \geq 1-\delta$,

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